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Artificial Boundary Conditions for the Linearized Compressible Navier–Stokes Equations

II. The Discrete Approach

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In the first part of this paper (*J. Comput. Phys.* **137**, 1, 1997), continuous artificial boundary conditions for the linearized compressible Navier–Stokes equations were proposed which were valid for small viscosities, high time frequencies, and long space wavelengths. In the present work, a new hierarchy of artificial boundary conditions is derived from the so-called "discrete" approach, which consists in working directly on the discretized equations, under the assumption of low time frequencies instead of small viscosities. The discrete artificial boundary conditions are implemented in 1D and 2D model problems and they compare quite well with the continuous artificial boundary conditions. Being self-sufficient by construction, they can be used as numerical boundary conditions and be coupled to schemes having arbitrary stencils. © 1998 Academic Press

INTRODUCTION

In order to compute in a bounded region a flow modeled by a problem formulated on an infinite domain, one often introduces an artificial boundary Γ and tries to write on the domain Ω bounded by Γ an initial boundary value problem whose solution is as close as possible to the solution of the original problem. When the solution of the mixed problem on Ω coincides with the restriction of the solution of the Cauchy problem, the boundary Γ is said to be *transparent*.

In general, the associated boundary condition, called the *transparent boundary condition*, is integral in time and space on the boundary and is usually replaced by local approximations, i.e., differential in time and space: the *artificial boundary conditions*.

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In the so-called “linear treatment,” the solution outside the artificial boundary is assumed to be a perturbation of a smooth steady state (often constant) about which the equations are linearized. The derivation and the analysis of the artificial boundary conditions is then performed on the linear equations.

The compressible Navier–Stokes equations belong to the class of *incompletely parabolic equations*. Laurence Halpern has proposeded in [3] a general method for deriving artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems, using the Fourier and Laplace transforms as essential tools after the equations have been linearized about a constant state. This method has been applied by the author to the compressible Navier–Stokes equations to obtain high order artificial boundary conditions, valid under the assumptions of small viscosities, high time frequencies, and long space wavelengths [5, 6].

There is another way of addressing the problem of the artificial boundary conditions. Introduced for the wave equations by Engquist and Majda [14] and also by Halpern [2], it consists in working directly on the discretized equations. This “discrete” approach has been successfully applied by the author in [5] to the compressible Navier–Stokes equations discretized by the explicit first order upwind scheme. The asymptotic expansions with respect to the viscosity are replaced by developments assuming low time frequencies. Other approaches for artificial boundary conditions have been proposed in Refs. [10–12]. A review on the subject can be found in [13].

This article, which is the continuation of [6], presents the main results of the work in [5], where the interested reader will find more details. In Section 1, the discrete transparent boundary condition for the negative half-space $i \leq 0$ is derived from the discrete transmission boundary conditions using a method quite similar to that employed in the continuous approach [6]. As in the continuous case, the discrete transparent boundary condition is integral in time and space on the boundary. In Section 2, the generalized eigenvalues and eigenvectors involved in the expression of the discrete transparent boundary condition are approximated by asymptotic expansions valid for low time frequencies and long space wavelengths. In Section 3, the results of Section 2 are used to build a hierarchy of discrete artificial boundary conditions which are local in time and space. In Section 4, 1D and 2D numerical results are presented and, in particular, the discrete artificial boundary conditions are compared to the continuous ones [6]. Finally, in Section 5, higher order discrete artificial boundary conditions are proposed.

Unlike the continuous artificial boundary conditions, the discrete artificial boundary conditions are self-sufficient: their number is always equal to the number of unknowns. Moreover, they can be used as numerical boundary conditions in the case where the continuous artificial boundary conditions need to be completed. Also, they can be coupled to schemes with arbitrary stencils since they are based on the definition of approximate values of the discrete solution outside the computational domain.

1. THE TRANSPARENT BOUNDARY CONDITION FOR THE NEGATIVE HALF-SPACE

Let $(u_{i,j})_{i,j\in\mathbb{Z}}$ be a family of vectors belonging to $\mathbb{R}^4$. We will denote $(g_{i,j})_{i\leq 0, j\in\mathbb{Z}}$ as the family $(u_{i,j})_{i\leq 0, j\in\mathbb{Z}}$ and $(d_{i,j})_{i\geq 0, j\in\mathbb{Z}}$ the sequence $(u_{i,j})_{i\geq 0, j\in\mathbb{Z}}$.

1.1. *The Transmission Boundary Conditions*

Let $(v_{i,j})_{i,j\in\mathbb{Z}}$ be an element of $(\mathbb{R}^4)^{\mathbb{Z}^2}$. It can be shown easily that $(u^n_{i,j})_{n\in\mathbb{N},i,j\in\mathbb{Z}}$ is solution of the discrete Cauchy problem,

$$\begin{cases} u^{n+1}_{i,j} = \sum_{\alpha,\beta=-1}^{1} A_{\alpha,\beta} u^n_{i+\alpha,j+\beta} & \forall n\in\mathbb{N}, \forall (i,j)\in\mathbb{Z}^2 \\ u^0_{i,j} = v_{i,j} & \forall (i,j)\in\mathbb{Z}^2 \end{cases} \tag{1.1}$$

if and only if $(g^n_{i,j})_{n\in\mathbb{N},i\le 0,j\in\mathbb{Z}}$ and $(d^n_{i,j})_{n\in\mathbb{N},i\ge 0,j\in\mathbb{Z}}$ solve

$$\begin{cases} g^{n+1}_{i,j} = \sum_{\alpha,\beta=-1}^{1} A_{\alpha,\beta} g^n_{i+\alpha,j+\beta} & \forall n\in\mathbb{N}, \forall i\le -1, \forall j\in\mathbb{Z} \\ g^0_{i,j} = v_{i,j} & \forall i\le 0, \forall j\in\mathbb{Z} \end{cases} \tag{1.2}$$

and

$$\begin{cases} d^{n+1}_{i,j} = \sum_{\alpha,\beta=-1}^{1} A_{\alpha,\beta} d^n_{i+\alpha,j+\beta} & \forall n\in\mathbb{N}, \forall i\ge 1, \forall j\in\mathbb{Z} \\ d^0_{i,j} = v_{i,j} & \forall i\ge 0, \forall j\in\mathbb{Z} \end{cases} \tag{1.3}$$

with the transmission boundary conditions

$$g^{n+1}_{0,j} = \sum_{\beta=-1}^{1} \left(\sum_{\alpha=-1}^{0} A_{\alpha,\beta} g^n_{\alpha,j+\beta} + A_{1,\beta} d^n_{1,j+\beta} \right) \qquad \forall n\in\mathbb{N}, \forall j\in\mathbb{Z} \tag{1.4}$$

$$g^n_{0,j} = d^n_{0,j} \qquad \forall n\in\mathbb{N}, \forall j\in\mathbb{Z}. \tag{1.5}$$

1.2. *The Solution in the Positive Half-Space*

The 2D linearized compressible Navier–Stokes equations have been presented in [6] and we use here the same notations. It is well known that the first order explicit upwind scheme, which is first-order consistent in both space and time, approaches the equation

$$\frac{\partial u}{\partial t} = A^{(1)} \frac{\partial u}{\partial x} + A^{(2)} \frac{\partial u}{\partial y} + \left(\nu P^{(1,1)} + \frac{\Delta x}{2} \left| A^{(1)} \right| \right) \frac{\partial^2 u}{\partial x^2} + \left(\nu P^{(2,2)} + \frac{\Delta y}{2} \left| A^{(2)} \right| \right) \frac{\partial^2 u}{\partial y^2} + 2\nu P^{(1,2)} \frac{\partial^2 u}{\partial x \partial y} \tag{1.6}$$

at first order in time and second order in space. In other words, Eq. (1.6) is the PDE equivalent to the first order explicit upwind scheme at first order in time and second order in space. By semi-discretization with respect to t and x, we obtain

$$u^{n+1}_i = \sum_{\alpha=-1}^{1} L_\alpha u^n_{i+\alpha}, \tag{1.7}$$

where the differential operators in y L_α are given by

$$L_{-1} = -\frac{\Delta t}{2\Delta x} A^{(1)} + \frac{\Delta t}{\Delta x^2} \left(\nu P^{(1,1)} + \frac{\Delta x}{2} \left| A^{(1)} \right| \right) - \nu \frac{\Delta t}{\Delta x} P^{(1,2)} \frac{d}{dy} \tag{1.8}$$

$$L_0 = I - 2\frac{\Delta t}{\Delta x^2}\left(\nu P^{(1,1)} + \frac{\Delta x}{2}\left|A^{(1)}\right|\right) + \Delta t A^{(2)}\frac{d}{dy} + \Delta t\left(\nu P^{(2,2)} + \frac{\Delta y}{2}\left|A^{(2)}\right|\right)\frac{d^2}{dy^2} = I + L_0' \tag{1.9}$$

$$L_1 = \frac{\Delta t}{2\Delta x}A^{(1)} + \frac{\Delta t}{\Delta x^2}\left(\nu P^{(1,1)} + \frac{\Delta x}{2}\left|A^{(1)}\right|\right) + \nu\frac{\Delta t}{\Delta x}P^{(1,2)}\frac{d}{dy}. \tag{1.10}$$

The initial value u^0 is assumed to have compact support in the negative half-space $i < 0$ and we consider the problem

$$d_i^{n+1}(y) = \sum_{\alpha=-1}^{1} L_\alpha d_{i+\alpha}^n(y) \qquad \forall\, n \in \mathbb{N}, \forall\, i \geq 1,\ \forall\, y \in \mathbb{R} \tag{1.11}$$

$$d_0^n(y) = g_n(y) \qquad \forall\, n \in \mathbb{N}, \forall\, y \in \mathbb{R} \tag{1.12}$$

$$d_i^0(y) = 0 \qquad \forall\, i \in \mathbb{N}, \forall\, y \in \mathbb{R} \tag{1.13}$$

with $(g_n)_{n\in\mathbb{N}}$ belonging to $\{[L^2(\mathbb{R})]^4\}^{\mathbb{N}}$.

The scheme defined by (1.11) can be viewed as acting on continuous functions of time taking their values in the Hilbert space $l^2(\mathbb{N}; [L^2(\mathbb{R})]^4)$ defined by

$$l^2(\mathbb{N}; [L^2(\mathbb{R})]^4) = \left\{(d_i) \in \{[L^2(\mathbb{R})]^4\}^{\mathbb{N}};\ \sum_{i=0}^{+\infty} |di|^2_{[L^2(\mathbb{R})]^4} < +\infty\right\}: \quad d_i(y, t + \Delta t) = \sum_{\alpha=-1}^{1} L_\alpha d_{i+\alpha}(y, t). \tag{1.14}$$

Denoting $\widehat{d_i}(\eta, s)$ as the Fourier–Laplace transform of $d_i(y, t)$ at points η and $s = \sigma + i\tau$, with $s > 0$, the relation (1.11) becomes

$$\widehat{L_{-1}}(\eta)\widehat{d_{i-1}}(\eta, s) + \left(\widehat{L_0'}(\eta) - \varepsilon I\right)\widehat{d_i}(\eta, s) + \widehat{L_1}(\eta)\widehat{d_{i+1}}(\eta, s) = 0. \tag{1.15}$$

The symbols $\widehat{L_{-1}}(\eta)$, $\widehat{L_0'}(\eta)$, and $\widehat{L_1}(\eta)$ of the differential operators L_{-1}, L_0', and L_1 are given by

$$\widehat{L_{-1}}(\eta) = -\frac{\Delta t}{2\Delta x}A^{(1)} + \frac{\Delta t}{\Delta x^2}\left(\nu P^{(1,1)} + \frac{\Delta x}{2}\left|A^{(1)}\right|\right) - \nu\frac{\Delta t}{\Delta x \Delta y}P^{(1,2)} i\eta\Delta y \tag{1.16}$$

$$\widehat{L_0'}(\eta) = -\frac{2\Delta t}{\Delta x^2}\left(\nu P^{(1,1)} + \frac{\Delta x}{2}\left|A^{(1)}\right|\right) + \frac{\Delta t}{\Delta y}A^{(2)} i\eta\Delta y + \frac{\Delta t}{\Delta y^2}\left(\nu P^{(2,2)} + \frac{\Delta y}{2}\left|A^{(2)}\right|\right)(i\eta\Delta y)^2 \tag{1.17}$$

$$\widehat{L_1}(\eta) = \frac{\Delta t}{2\Delta x}A^{(1)} + \frac{\Delta t}{\Delta x^2}\left(\nu P^{(1,1)} + \frac{\Delta x}{2}\left|A^{(1)}\right|\right) + \frac{\nu\Delta t}{\Delta x \Delta y}P^{(1,2)} i\eta\Delta y, \tag{1.18}$$

where $\varepsilon = z - 1$ and $z = e^{s\Delta t} = e^{\sigma\Delta t}e^{i\tau\Delta t}$.

According to [4], the general solution in $l^2(\mathbb{N}; [L^2(\mathbb{R})]^4)$ of the difference equation (1.15) reads

$$\widehat{d_i}(\eta, s) = \sum_{j/|\rho_j|<1} P_j(i, \eta, s)[\rho_j(\eta, s)]^i, \tag{1.19}$$

where the $\rho_j(\eta, s)$ are the roots with $|\rho_j(\eta, s)| < 1$ of the algebraic equation of degree 8

$$\det\left(\widehat{L_{-1}}(\eta)\rho^{-1} + \widehat{L_0'}(\eta) + \widehat{L_1}(\eta)\rho - \varepsilon I\right) = 0 \tag{1.20}$$

and where the $P_j(i, \eta, s)$ are polynomials in i with coefficients in $\mathbb{C}^4$, the degree of P_j being one less than the multiplicity of ρ_j.

PROPOSITION 1. *For each complex number z with $|z| > 1$, $\eta(z) > 0$ exists such that $\forall\, |\eta| < \eta(z)$, Eq.* (1.20) *admits four roots with $|\rho| < 1$, the four remaining verifying $|\rho| > 1$.*

Proof. See [5].

Let us now consider the following generalized eigenvalue problem: find $(\rho, \Phi) \in \mathbb{C} \times \mathbb{C}^4$ solving

$$\left(\sum_{\alpha=-1}^{1} \widehat{L_\alpha}(\eta)\rho^\alpha - zI\right)\Phi = 0. \tag{1.21}$$

For (ρ, Φ) solution of (1.21), ρ and Φ will be respectively called a *generalized eigenvalue* and a *generalized eigenvector* (related to the generalized eigenvalue ρ). We have the

THEOREM 1. *We assume that $r > 0$ exists such that for each complex number z with $|z| > 1$, verifying $|z - 1| < r$ and for each real number η with $|\eta| < \eta(z)$ (see Proposition* 1), *the generalized eigenvectors Φ^j, $j = 1, \ldots, 4$ associated to the generalized eigenvalues ρ with $|\rho| < 1$ are linearly independent.*

Then, the solution in $l^2(\mathbb{N}; [L^2(\mathbb{R})]^4)$ of the problem (1.11)–(1.13) *reads, in terms of variables η and s,*

$$\widehat{d_i}(\eta, s) = \sum_{j=1}^{4} \lambda_j(\eta, s)[\rho_j(\eta, s)]^i \Phi^j(\eta, s), \tag{1.22}$$

where (ρ_j, Φ^j) are solutions of Eq. (1.21), *the coefficients λ_j being determined by the boundary conditions* (1.12).

Proof. Following (1.19), the set of the solutions in $l^2(\mathbb{N}; [L^2(\mathbb{R})]^4)$ of the difference equation (1.15) is a four dimensional vector space. The Φ^j, $j = 1, \ldots, 4$, being linearly independant, the elementary solutions $(\rho_j)^i \Phi^j$, $j = 1, \ldots, 4$, are also linearly independant thus forming a basis of that space.

1.3. *The Transparent Boundary Condition*

The coefficients λ_j in the above theorem solve the fourth order linear system

$$\sum_{j=1}^{4} \lambda_j \Phi^j = \hat{g}(\eta, s). \tag{1.23}$$

With M the 4×4 matrix defined by $M_{jk} = \Phi_j^k$ and $N = M^{-1}$ its inverse, they are given by $\lambda_j = \sum_{k=1}^{4} N_{jk}\, \widehat{g_k}$.

Let $g_0^n(y)$ be the projection of family $(g_{0,j}^n)$ on the set of continuous functions, linear in each segment $[y_j, y_{j+1}]$. If we choose $g_n(y) = g_0^n(y)$, the transmission condition (1.4) leads to:

THEOREM 2. *The transparent boundary condition at $i=0$ for the negative half-space is*

$$g_{0,j}^{n+1} = \sum_{\beta=-1}^{1} \left\{ \sum_{\alpha=-1}^{0} A_{\alpha,\beta} g_{\alpha,j+\beta}^{n} + A_{1,\beta} \int_{\mathbb{R}^2} e^{i\eta(y_j+\beta\Delta y)} e^{(\sigma+i\tau)t_n} \times \left[\sum_{j,k=1}^{4} N_{jk} \widehat{(g_0)}_k (\rho_j)^1 \Phi^j \right] (\eta, \sigma+i\tau)\, d\eta\, d\tau \right\}. \tag{1.24}$$

2. GENERALIZED EIGENVALUES AND EIGENVECTORS

As in the continuous case, the generalized eigenvalues and eigenvectors defined by (1.21) are non-rational functions of variables z and η and the transparent boundary condition (1.24) is thus integral with respect to t and y. In order to obtain boundary conditions which are local in time and space, we will develop the generalized eigenvalues and eigenvectors at first order with respect to the parameters $\varepsilon = z - 1$ and $\eta^* = \eta\Delta y/\varepsilon$ assumed small.

2.1. *General Considerations*

We will not make any approximation with respect to the viscosity ν because the quantities $\nu\Delta t/\Delta x^2$, $\nu\Delta t/\Delta y^2$, and $\nu\Delta t/(\Delta x\Delta y)$ may be big even if $\nu \ll 1$ and we will work with the characteristic variables w of matrix $A^{(1)}$. They are defined by $w = \mathcal{P}^{(1)^{-1}} u$ and solve the equation

$$\frac{\partial}{\partial t} w = \Lambda^{(1)} \frac{\partial}{\partial x} w + \mathcal{A}^{(2)} \frac{\partial}{\partial y} w + \nu B^{(1,1)} \frac{\partial^2}{\partial x^2} w + \nu B^{(2,2)} \frac{\partial^2}{\partial y^2} w + 2\nu B^{(1,2)} \frac{\partial^2}{\partial x \partial y} w \tag{2.1}$$

with

$$\mathcal{A}^{(2)} = \begin{pmatrix} -\overline{V}_2 & -1/2 & 0 & 0 \\ -\overline{C}^2 & -\overline{V}_2 & 0 & \overline{C}^2 \\ 0 & 0 & -\overline{V}_2 & 0 \\ 0 & 1/2 & 0 & -\overline{V}_2 \end{pmatrix} \tag{2.2}$$

$$B^{(1,1)} = \begin{pmatrix} \frac{2}{3} + \frac{\gamma-1}{2\,\mathrm{Pr}} & 0 & \frac{1}{2\,\mathrm{Pr}} & \frac{2}{3} - \frac{\gamma-1}{2\,\mathrm{Pr}} \\ 0 & 1 & 0 & 0 \\ \frac{\gamma-1}{\mathrm{Pr}} & 0 & \frac{1}{\mathrm{Pr}} & -\frac{\gamma-1}{\mathrm{Pr}} \\ \frac{2}{3} - \frac{\gamma-1}{2\,\mathrm{Pr}} & 0 & -\frac{1}{2\,\mathrm{Pr}} & \frac{2}{3} + \frac{\gamma-1}{2\,\mathrm{Pr}} \end{pmatrix} \tag{2.3}$$

$$B^{(2,2)} = \begin{pmatrix} \frac{1}{2} + \frac{\gamma-1}{2\,\mathrm{Pr}} & 0 & \frac{1}{2\,\mathrm{Pr}} & \frac{1}{2} - \frac{\gamma-1}{2\,\mathrm{Pr}} \\ 0 & \frac{4}{3} & 0 & 0 \\ \frac{\gamma-1}{\mathrm{Pr}} & 0 & \frac{1}{\mathrm{Pr}} & -\frac{\gamma-1}{\mathrm{Pr}} \\ \frac{1}{2} - \frac{\gamma-1}{2\,\mathrm{Pr}} & 0 & -\frac{1}{2\,\mathrm{Pr}} & \frac{1}{2} + \frac{\gamma-1}{2\,\mathrm{Pr}} \end{pmatrix} \tag{2.4}$$

$$B^{(1,2)} = \frac{1}{6}\begin{pmatrix} 0 & \frac{1}{2\overline{C}} & 0 & 0 \\ \overline{C} & 0 & 0 & \overline{C} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\overline{C}} & 0 & 0 \end{pmatrix} \tag{2.5}$$

$$\left|\mathcal{A}^{(2)}\right| = \begin{pmatrix} \frac{1}{2}(|\overline{V}_2| + M_2) & \frac{m_2}{2\overline{C}} & 0 & \frac{1}{2}(|\overline{V}_2| - M_2) \\ \overline{C}m_2 & M_2 & 0 & -\overline{C}m_2 \\ 0 & 0 & |\overline{V}_2| & 0 \\ \frac{1}{2}(|\overline{V}_2| - M_2) & -\frac{m_2}{2\overline{C}} & 0 & \frac{1}{2}(|\overline{V}_2| + M_2) \end{pmatrix} \tag{2.6}$$

$$m_2 = \operatorname{sgn}(\overline{V}_2)\min(|\overline{V}_2|, \overline{C}), \qquad M_2 = \max(|\overline{V}_2|, \overline{C}). \tag{2.7}$$

We choose the notations

$$\rho = \rho_{00} + i\overline{\eta}\rho_{01} + \varepsilon\rho_{10} + (i\overline{\eta})^2\rho_{02} + i\overline{\eta}\varepsilon\rho_{11} + \varepsilon^2\rho_{20} + O(\varepsilon^3) + O(\overline{\eta}^3), \tag{2.8}$$

where we have introduced the non-dimensional variable $\overline{\eta} = \eta\Delta y = \varepsilon\eta^*$. The solutions (ρ, Φ) of the generalized eigenvalue problem (1.21) verify $\mathcal{M}(\rho, \varepsilon, \overline{\eta})\Phi = 0$ with

$$\mathcal{M}(\rho, \varepsilon, \overline{\eta}) = \widehat{L_{-1}}^{00} + i\overline{\eta}\widehat{L_{-1}}^{01} + \rho\big(\widehat{L_0'}^{00} + i\overline{\eta}\widehat{L_0'}^{01} + (i\overline{\eta})^2\widehat{L_0'}^{02} - \varepsilon I\big) + \rho^2\big(\widehat{L_1}^{00} + i\overline{\eta}\widehat{L_1}^{01}\big) \tag{2.9}$$

and

$$\widehat{L_{-1}}^{00} = -\frac{\Delta t}{2\Delta x}\big(\Lambda^{(1)} - \left|\Lambda^{(1)}\right|\big) + \frac{\nu\Delta t}{\Delta x^2}B^{(1,1)}, \qquad \widehat{L_{-1}}^{01} = -\frac{\nu\Delta t}{\Delta x\Delta y}B^{(1,2)} \tag{2.10}$$

$$\widehat{L_0'}^{00} = -\frac{\Delta t}{\Delta x}\left|\Lambda^{(1)}\right| - \frac{2\nu\Delta t}{\Delta x^2}B^{(1,1)}, \qquad \widehat{L_0'}^{01} = \frac{\Delta t}{\Delta y}\mathcal{A}^{(2)},$$
$$\widehat{L_0'}^{02} = \frac{\Delta t}{2\Delta y}\left|\mathcal{A}^{(2)}\right| + \frac{\nu\Delta t}{\Delta y^2}B^{(2,2)} \tag{2.11}$$

$$\widehat{L_1}^{00} = \frac{\Delta t}{2\Delta x}\big(\Lambda^{(1)} + \left|\Lambda^{(1)}\right|\big) + \frac{\nu\Delta t}{\Delta x^2}B^{(1,1)}, \qquad \widehat{L_1}^{01} = \frac{\nu\Delta t}{\Delta x\Delta y}B^{(1,2)}. \tag{2.12}$$

Injecting the asymptotic expansion (2.8) in the expression of matrix $\mathcal{M}$, we obtain

$$\mathcal{M} = \mathcal{M}_{00} + i\overline{\eta}\mathcal{M}_{01} + \varepsilon\mathcal{M}_{10} + (i\overline{\eta})^2\mathcal{M}_{02} + i\overline{\eta}\varepsilon\mathcal{M}_{11} + \varepsilon^2\mathcal{M}_{20} + O(\varepsilon^3) + O(\overline{\eta}^3) \tag{2.13}$$

with

$$\mathcal{M}_{00} = \widehat{L_{-1}}^{00} + \rho_{00}\widehat{L_0'}^{00} + \rho_{00}^2\widehat{L_1}^{00} \tag{2.14}$$

$$\mathcal{M}_{01} = \widehat{L_{-1}}^{01} + \rho_{00}\widehat{L_0'}^{01} + \rho_{00}^2\widehat{L_1}^{01} + \rho_{01}\big(\widehat{L_0'}^{00} + 2\rho_{00}\widehat{L_1}^{00}\big)$$
$$\mathcal{M}_{10} = \rho_{10}\big(\widehat{L_0'}^{00} + 2\rho_{00}\widehat{L_1}^{00}\big) - \rho_{00}I \tag{2.15}$$

$$\left\{\begin{aligned}\mathcal{M}_{02} &= \rho_{02}\big(\widehat{L_0'}^{00} + 2\rho_{00}\widehat{L_1}^{00}\big) + \rho_{01}^2\widehat{L_1}^{00} + \rho_{01}\big(\widehat{L_0'}^{01} + 2\rho_{00}\widehat{L_1}^{01}\big) + \rho_{00}\widehat{L_0'}^{02}\\ \mathcal{M}_{11} &= \rho_{11}\big(\widehat{L_0'}^{00} + 2\rho_{00}\widehat{L_1}^{00}\big) + \rho_{10}\big(\widehat{L_0'}^{01} + 2\rho_{00}\widehat{L_1}^{01}\big) + 2\rho_{01}\rho_{10}\widehat{L_1}^{00} - \rho_{01}I\\ \mathcal{M}_{20} &= \rho_{20}\big(\widehat{L_0'}^{00} + 2\rho_{00}\widehat{L_1}^{00}\big) + \rho_{10}^2\widehat{L_1}^{00} - \rho_{10}I.\end{aligned}\right. \tag{2.16}$$

Matrix $\mathcal{M}_{00}$ can easily be put under the form

$$\mathcal{M}_{00} = (1-\rho_{00})[\mathcal{D}_0 + (1-\rho_{00})(\mathcal{D}_1 + \overline{\nu}B)] \tag{2.17}$$

with

$$\alpha_1 = (\overline{V}_1 + \overline{C})\frac{\Delta t}{\Delta x}, \qquad \alpha_2 = \alpha_3 = \overline{V}_1\frac{\Delta t}{\Delta x}, \qquad \alpha_4 = (\overline{V}_1 - \overline{C})\frac{\Delta t}{\Delta x} \tag{2.18}$$

$$\alpha_i^+ = \max(0, \alpha_i), \qquad \alpha_i^- = \max(0, -\alpha_i), \quad i = 1, \ldots, 4 \tag{2.19}$$

$$\overline{\nu} = \frac{\nu\Delta t}{\Delta x^2} \tag{2.20}$$

$$\begin{gathered}\mathcal{D}_{-1} = \operatorname{diag}(\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_4^+), \qquad \mathcal{D}_0 = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4),\\ \mathcal{D}_1 = \operatorname{diag}(\alpha_1^-, \alpha_2^-, \alpha_3^-, \alpha_4^-), \qquad B = B^{(1,1)}\end{gathered} \tag{2.21}$$

and we see that the algebraic equation of degree 8 $\det(\mathcal{M}_{00}) = 0$ admits the quadruple root $\rho_{00} = 1$. We will thus distinguish the generalized eigenvalues whose limit is 1 as $(\varepsilon, \eta^*) \to (0, 0)$ from the others. In the sequel, they will be named respectively "generalized eigenvalues of the first kind" and "generalized eigenvalues of the second kind."

2.2. *Generalized Eigenvalues and Eigenvectors of the First Kind*

For the generalized eigenvalues of the first kind, the matrix $\mathcal{M}_{00}$ vanishes and we have

$$\mathcal{M} = \varepsilon\big\{\mathcal{M}_{10} + i\eta^*\mathcal{M}_{01} + \varepsilon\mathcal{M}_{20} + i\eta^*\varepsilon\mathcal{M}_{11} + (i\eta^*)^2\varepsilon\mathcal{M}_{02}\big\} + O(\varepsilon^3) + O(\eta^{*3}). \tag{2.22}$$

ρ_{10}, ρ_{01}, and ρ_{20} are given respectively by equations

$$\det(\mathcal{M}_{10}) = 0 \tag{2.23}$$

$$\det{}'(\mathcal{M}_{10}) \cdot \mathcal{M}_{01} = 0 \tag{2.24}$$

$$\det{}'(\mathcal{M}_{10}) \cdot \mathcal{M}_{20} = 0. \tag{2.25}$$

The associated generalized eigenvectors are expanded under the form

$$\Phi = \Phi_{00} + i\eta^*\Phi_{01} + \varepsilon\Phi_{10} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.26}$$

and the terms Φ_{00}, Φ_{01}, and Φ_{10} are determined by solving the linear systems

$$\mathcal{M}_{10}\Phi_{00} = 0 \tag{2.27}$$

$$\mathcal{M}_{10}\Phi_{01} = -\mathcal{M}_{01}\Phi_{00} \tag{2.28}$$

and

$$\mathcal{M}_{10}\Phi_{10} = -\mathcal{M}_{20}\Phi_{00}. \tag{2.29}$$

We finally obtain

$$\rho_i = 1 + i\eta \frac{-\bar{V}_2(\Delta t/\Delta y)}{\alpha_i} - \frac{1}{\alpha_i}\varepsilon + \frac{1}{\alpha_i^2}\left[1 + \frac{1}{\alpha_i}(\bar{\nu} B_{ii} + \alpha_i^-)\right]\varepsilon^2 + O(\varepsilon^3) + O(\eta^{*2}) \tag{2.30}$$

$$\Phi^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{\varepsilon\bar{\nu}}{\alpha_1}\begin{pmatrix}0\\ B_{21}/(\alpha_1-\alpha_2)\\ B_{31}/(\alpha_1-\alpha_3)\\ B_{41}/(\alpha_1-\alpha_4)\end{pmatrix} + i\eta^*\frac{\Delta t}{\Delta y}\begin{pmatrix}0\\ -\bar{C}(\bar{V}_1+\bar{C})\\ 0\\ 0\end{pmatrix} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.31}$$

$$\Phi^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \varepsilon\mathbf{0} + i\eta^*\frac{\Delta t}{\Delta y}\begin{pmatrix}\frac{1}{2}\frac{\bar{V}_1}{\bar{C}}\\ 0\\ 0\\ \frac{1}{2}\frac{\bar{V}_1}{\bar{C}}\end{pmatrix} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.32}$$

$$\Phi^3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \frac{\varepsilon\bar{\nu}}{\alpha_3}\begin{pmatrix}B_{13}/(\alpha_3-\alpha_1)\\ B_{23}/(\alpha_3-\alpha_2)\\ 0\\ B_{43}/(\alpha_3-\alpha_4)\end{pmatrix} + i\eta^*\mathbf{0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.33}$$

$$\Phi^4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} + \frac{\varepsilon\bar{\nu}}{\alpha_4}\begin{pmatrix}B_{14}/(\alpha_4-\alpha_1)\\ B_{24}/(\alpha_4-\alpha_2)\\ B_{34}/(\alpha_4-\alpha_3)\\ 0\end{pmatrix} + i\eta^*\frac{\Delta t}{\Delta y}\begin{pmatrix}0\\ -\bar{C}(\bar{V}_1-\bar{C})\\ 0\\ 0\end{pmatrix} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.34}$$

and we will only consider the (ρ_i, Φ^i) such that $\alpha_i > 0$ for which it is shown in [5] that $|1 - (1/\alpha_i)\varepsilon| < 1$.

2.3. *Generalized Eigenvalues and Eigenvectors of the Second Kind*

ρ_{00} is a root of the algebraic equation of degree 4 $\det[\mathcal{D}_0 + (1-\rho_{00})(\mathcal{D}_1 + \bar{\nu}B)] = 0$, which admits the obvious root $\rho_{00} = 1 + \alpha_2/(\bar{\nu} + \alpha_2^-)$, and ρ_{10} is given by $\det'(\mathcal{M}_{00}) \cdot \mathcal{M}_{10} = 0$.

If we set $\Phi = \Phi_{00} + \varepsilon\Phi_{10} + O(\varepsilon^2) + O(\bar{\eta})$, the vectors Φ_{00} and Φ_{10} are respectively given by $\mathcal{M}_{00}\Phi_{00} = 0$ and $\mathcal{M}_{00}\Phi_{10} = -\mathcal{M}_{10}\Phi_{00}$ and we obtain

$$\rho_i = \left(1 + \frac{1}{\bar{\nu}\chi_i}\right)\left(1 + \varepsilon\frac{\Delta_1 + \Delta_3 + \Delta_4}{\alpha_1\Delta_1 + \alpha_3\Delta_3 + \alpha_4\Delta_4}\right) + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.35}$$

$$\rho_4 = \left(1 + \frac{\alpha_2}{\bar{\nu} + \alpha_2^-}\right)\left(1 + \varepsilon\frac{1}{\alpha_2}\right) + O(\varepsilon^2) + O(\eta^{*2}). \tag{2.36}$$

χ_1, χ_2, and χ_3 denote the roots of the algebraic equation of degree 3 with real coefficients

$$\chi^3 + a_2\chi^2 + a_1\chi + a_0 = 0 \tag{2.37}$$

with

$$a_2 = -\left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_4^-}{\bar{\nu}}\right)\frac{1}{\alpha_4} - \left(\frac{1}{\mathrm{Pr}} + \frac{\alpha_3^-}{\bar{\nu}}\right)\frac{1}{\alpha_3} - \left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_1^-}{\bar{\nu}}\right)\frac{1}{\alpha_1} \tag{2.38}$$

$$\begin{aligned} a_1 = {} & \frac{1}{\alpha_1\alpha_3}\left[\left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_1^-}{\bar{\nu}}\right)\left(\frac{1}{\mathrm{Pr}} + \frac{\alpha_3^-}{\bar{\nu}}\right) - \frac{\gamma - 1}{2\,\mathrm{Pr}^2}\right] \\ & + \frac{1}{\alpha_1\alpha_4}\left[\left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_1^-}{\bar{\nu}}\right)\left(\frac{8}{3} + \frac{\alpha_1^-}{\bar{\nu}} + \frac{\alpha_4^-}{\bar{\nu}}\right) - \left(\frac{4}{3} + \frac{\alpha_1^-}{\bar{\nu}}\right)^2\right] \\ & + \frac{1}{\alpha_3\alpha_4}\left[\left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_4^-}{\bar{\nu}}\right)\left(\frac{1}{\mathrm{Pr}} + \frac{\alpha_3^-}{\bar{\nu}}\right) - \frac{\gamma - 1}{2\,\mathrm{Pr}^2}\right] \end{aligned} \tag{2.39}$$

$$\begin{aligned} a_0 = {} & \frac{1}{\alpha_1\alpha_3\alpha_4}\left\{\left(\frac{1}{\mathrm{Pr}} + \frac{\alpha_3^-}{\bar{\nu}}\right)\left[\left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}}\right)^2 - \left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_1^-}{\bar{\nu}}\right)\right.\right. \\ & \left.\left. \times \left(\frac{2}{3} + \frac{\gamma - 1}{2\,\mathrm{Pr}} + \frac{\alpha_4^-}{\bar{\nu}}\right)\right] + \frac{\gamma - 1}{2\,\mathrm{Pr}^2}\left(\frac{8}{3} + \frac{\alpha_1^-}{\bar{\nu}} + \frac{\alpha_4^-}{\bar{\nu}}\right)\right\} \end{aligned} \tag{2.40}$$

and Δ_1, Δ_3, and Δ_4 are defined by

$$\Delta_1 = \{\alpha_3 + (1 - \rho_{00})[\bar{\nu}B_{33} + \alpha_3^-]\}\{\alpha_4 + (1 - \rho_{00})[\bar{\nu}B_{44} + \alpha_4^-]\} - (1 - \rho_{00})^2\bar{\nu}^2 B_{43}B_{34} \tag{2.41}$$

$$\Delta_3 = \{\alpha_4 + (1 - \rho_{00})[\bar{\nu}B_{44} + \alpha_4^-]\}\{\alpha_1 + (1 - \rho_{00})[\bar{\nu}B_{11} + \alpha_1^-]\} - (1 - \rho_{00})^2\bar{\nu}^2 B_{14}B_{41} \tag{2.42}$$

$$\Delta_4 = \{\alpha_1 + (1 - \rho_{00})[\bar{\nu}B_{11} + \alpha_1^-]\}\{\alpha_3 + (1 - \rho_{00})[\bar{\nu}B_{33} + \alpha_3^-]\} - (1 - \rho_{00})^2\bar{\nu}^2 B_{31}B_{13}. \tag{2.43}$$

For the associated generalized eigenvectors, we have

$$\begin{aligned} \Phi^i = {} & \begin{bmatrix} \frac{4}{3}\frac{1}{2\,\mathrm{Pr}} + \left(\frac{2}{3} - \frac{\gamma-1}{2\,\mathrm{Pr}}\right)\left(\frac{\alpha_3^-}{\bar{\nu}} - \alpha_3\chi_i\right) \\ 0 \\ \frac{\gamma - 1}{\mathrm{Pr}}\left(\alpha_1\chi_i - \frac{4}{3} - \frac{\alpha_1^-}{\bar{\nu}}\right) \\ \left(\frac{2}{3} + \frac{\alpha_1^-}{\bar{\nu}} - \alpha_1\chi_i\right)\left(\alpha_3\chi_i - \frac{1}{\mathrm{Pr}} - \frac{\alpha_3^-}{\bar{\nu}}\right) + \frac{\gamma - 1}{2\,\mathrm{Pr}}\left(\alpha_3\chi_i - \frac{\alpha_3^-}{\bar{\nu}}\right) \end{bmatrix} \\ & + \varepsilon \begin{bmatrix} \frac{((\gamma - 1)/\mathrm{Pr})b_1 + (2/3 - ((\gamma - 1)/2\,\mathrm{Pr}))b_3}{\Delta} \\ \frac{b_2}{\alpha_2 + (1 - \rho_{00})(\bar{\nu} + \alpha_2^-)} \\ 0 \\ \frac{((\gamma - 1)/\mathrm{Pr})b_1 - (2/3 + ((\gamma - 1)/2\,\mathrm{Pr}) + \alpha_1^-/\bar{\nu} - \alpha_1\chi_i)b_3}{\Delta} \end{bmatrix} + O(\varepsilon^2) + O(\eta^{*2}), \qquad i = 1, 2, 3 \end{aligned} \tag{2.44}$$

$$\Phi^4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \varepsilon \mathbf{0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{2.45}$$

with

$$\Delta = \frac{\gamma - 1}{\Pr}\left[\alpha_1 + (1 - \rho_{00})\left(\frac{4}{3}\overline{\nu} + \alpha_1^-\right)\right], \tag{2.46}$$

$$b = \frac{1}{1 - \rho_{00}}(\rho_{00} I - \rho_{10}\mathcal{D}_0)\Phi_{00}. \tag{2.47}$$

For $i \leq 3$, we use ρ_i when $\chi_i < 0$ whereas ρ_4 is kept in the case $\alpha_2 < 0$.

2.4. *Interpretation of the Assumptions Underlying the Asymptotic Expansions*

Taking the Fourier–Laplace transform of Eq. (1.11) is equivalent to looking for solutions of the form

$$d_i(y, t) = e^{\sigma t} e^{i(\eta y + \tau t)} \widehat{d_i}(\eta, \sigma + i\tau) \tag{2.48}$$

which have a sinusoidal behaviour in time and space, with respective periods $T = \frac{2\pi}{|\tau|}$ and $\lambda = \frac{2\pi}{|\eta|}$. We choose a time step Δt and a space step Δy. Only those waves for which $T \gg \Delta t$, or equivalently $|\tau|\Delta t \ll 2\pi$, are well seen by the time discretization. If we consider that $e^{i\tau\Delta t} - 1$ is equivalent to $i\tau\Delta t$ as τ tends to 0, we recover the assumption $|\varepsilon| \ll 1$ small behind the previous asymptotic expansions.

Moreover, the phase velocity V_φ of the sinusoidal plane wave $e^{i(\eta y + \tau t)}\widehat{d_i}$ is equal to $-\tau/\eta$ because a perturbation which is in y at time t comes in $y - \frac{\tau}{\eta}\, dt$ at time $t + dt$ as we have $\tau(t + dt) + \eta(y - \frac{\tau}{\eta}\, dt) = \tau t + \eta y$. The assumption $|\eta^*| = |\frac{\eta \Delta y}{\varepsilon}| \ll 1$ is then equivalent to $|V_\varphi| \gg \frac{\Delta y}{\Delta t}$ which means that the phase velocity of the wave is much bigger than the numerical velocity.

Both assumptions $|\varepsilon| \ll 1$ and $|\eta^*| \ll 1$ imply that $|\eta \Delta y| = |\varepsilon \eta^*| \ll 1$ which is equivalent to $\lambda \gg \Delta y$ and means that the wave is also well seen by the space discretization.

3. APPROXIMATION OF THE TRANSPARENT BOUNDARY CONDITION

On the basis of the asymptotic expansions of the generalized eigenvalues and eigenvectors, we will approximate the right-hand side in relation (1.22), written at $i = 1$, by a polynomial in the variables ε and η^*.

3.1. *General Considerations*

We will call discrete artificial boundary conditions of order (0, 0) (resp. (1, 0)) (resp. (1, 1)) the boundary conditions corresponding to the representation

$$\widehat{d_1} = (\widehat{d_1})_{00} \qquad (\text{resp. } \widehat{d_1} = (\widehat{d_1})_{00} + \varepsilon(\widehat{d_1})_{10}) \ (\text{resp. } \widehat{d_1} = (\widehat{d_1})_{00} + \eta^*(\widehat{d_1})_{01} + \varepsilon(\widehat{d_1})_{10})$$

with obvious notations. By an inverse Fourier–Laplace transform, we will obtain an equation which, after discretization with respect to the space variable y, will be used to compute the ficticious values $d^n_{1,j}$, thus allowing the scheme to be applied up to the boundary points (x_0, y_j).

More precisely, the matrix M defined in Subsection 1.3 admits the development $M = M_{00} + i\eta^* M_{01} + \varepsilon M_{10} + O(\varepsilon^2) + O(\eta^{*2})$. Its inverse N is given by

$$N = N_{00} + i\eta^* N_{01} + \varepsilon N_{10} + O(\varepsilon^2) + O(\eta^{*2}) \tag{3.1}$$

with $N_{00} = (M_{00})^{-1}$, $N_{01} = -N_{00}M_{01}N_{00}$, $N_{10} = -N_{00}M_{10}N_{00}$ and we have

$$\begin{aligned}\widehat{d_1} = \sum_{j=1}^{4}(\widehat{g_0})_j \Bigg\{ &\sum_{i=1}^{4}(N_{ij})_{00}(\rho_i)_{00}\Phi^i_{00} + i\eta^* \sum_{i=1}^{4}\left[(N_{ij})_{01}(\rho_i)_{00}\Phi^i_{00} + (N_{ij})_{00}(\rho_i)_{00}\Phi^i_{01}\right] \\ &+ \varepsilon \sum_{i=1}^{4}\left[(N_{ij})_{10}(\rho_i)_{00}\Phi^i_{00} + (N_{ij})_{00}(\rho_i)_{10}\Phi^i_{00} + (N_{ij})_{00}(\rho_i)_{00}\Phi^i_{10}\right]\Bigg\} \\ &+ O(\varepsilon^2) + O(\eta^{*2})\end{aligned} \tag{3.2}$$

which can be written under the condensed form

$$\widehat{d_1} = (\mathcal{A}_{00} + i\eta^*\mathcal{A}_{01} + \varepsilon\mathcal{A}_{10})\widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}). \tag{3.3}$$

3.2. *Approximation of Order* (1, 0)

At order 0 with respect to η^*, the relation (3.3) becomes $\widehat{d_1} = (\mathcal{A}_{00} + \varepsilon\mathcal{A}_{10})\widehat{g_0} + O(\varepsilon^2) + O(\eta^*)$. We have successively

$$\widehat{d_1} = \mathcal{A}_{00}\big(I + \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\big)\widehat{g_0} + O(\varepsilon^2) + O(\eta^*) \tag{3.4}$$

$$\big(I + \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\big)^{-1}\mathcal{A}_{00}^{-1}\widehat{d_1} = \widehat{g_0} + O(\varepsilon^2) + O(\eta^*) \tag{3.5}$$

$$\big(I - \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\big)\mathcal{A}_{00}^{-1}\widehat{d_1} = \widehat{g_0} + O(\varepsilon^2) + O(\eta^*) \tag{3.6}$$

$$\varepsilon\widehat{d_1} = \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\widehat{d_1} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{00}\widehat{g_0} + O(\varepsilon^2) + O(\eta^*) \tag{3.7}$$

and we finally obtain

$$d^{n+1}_{1,j} = \big(\mathcal{A}_{00}\mathcal{A}_{10}^{-1} + I\big)d^n_{1,j} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{00}g^n_{0,j}. \tag{3.8}$$

3.3. *Approximation of Order* (1, 1)

Following (3.3), we have successively

$$\widehat{d_1} - i\eta^*\mathcal{A}_{01}\widehat{g_0} = \mathcal{A}_{00}\big(I + \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\big)\widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{3.9}$$

$$\big(I + \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\big)^{-1}\mathcal{A}_{00}^{-1}\big(\widehat{d_1} - i\eta^*\mathcal{A}_{01}\widehat{g_0}\big) = \widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{3.10}$$

$$\big(\mathcal{A}_{00}^{-1} - \varepsilon\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\mathcal{A}_{00}^{-1}\big)\big(\widehat{d_1} - i\eta^*\mathcal{A}_{01}\widehat{g_0}\big) = \widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{3.11}$$

$$\mathcal{A}_{00}^{-1}\mathcal{A}_{10}\mathcal{A}_{00}^{-1}\varepsilon\widehat{d_1} = \mathcal{A}_{00}^{-1}\widehat{d_1} - i\eta^*\mathcal{A}_{00}^{-1}\mathcal{A}_{01}\widehat{g_0} - \widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}) \tag{3.12}$$

$$\varepsilon\widehat{d_1} = \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\widehat{d_1} - i\eta^*\mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{01}\widehat{g_0} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{00}\widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}). \tag{3.13}$$

After multiplication by ε and inverse Fourier–Laplace transform, we obtain

$$d_1^{n+1}(y) = 2d_1^n(y) - d_1^{n-1}(y) + \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\left(d_1^n - d_1^{n-1}\right)(y) - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{01}\Delta y \frac{d}{dy} g_0^{n-1}(y)$$
$$- \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{00}\left(g_0^n - g_0^{n-1}\right)(y). \tag{3.14}$$

Approximating the y-derivative by a second order centered finite difference, we end up with

$$d_{1,j}^{n+1} = 2d_{1,j}^n - d_{1,j}^{n-1} + \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\left(d_{1,j}^n - d_{1,j}^{n-1}\right) - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{01}\frac{1}{2}\left(g_{0,j+1}^{n-1} - g_{0,j-1}^{n-1}\right)$$
$$- \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{00}\left(g_{0,j}^n - g_{0,j}^{n-1}\right). \tag{3.15}$$

3.4. *Interpretation of the Discrete Artificial Boundary Conditions of Order* (0, 0) *in the Supersonic and Subsonic Outflow Cases*

In the supersonic outflow case, the α_i's are all strictly positive and we have $(\rho_i)_{00} = 1$, $i = 1, \ldots, 4$, $\Phi_{00}^i = \mathbf{e}_i$, $i = 1, \ldots, 4$, and $N_{00} = I$, where $\mathbf{e}_i$ denotes the ith vector of the canonical basis of $\mathbb{R}^4$. The matrix $\mathcal{A}_{00}$ is then equal to the identy matrix and we obtain $d_{1,j}^{n+1} = g_{0,j}^{n+1}$, i.e., zeroth order extrapolation of the characteristic variables of matrix $A^{(1)}$, which makes sense as they all propagate towards the positive x values.

In the subsonic outflow case, only α_1, α_2, and α_3 are strictly positive and we have

$$(\rho_i)_{00} = 1, i = 1, 2, 3, \qquad (\rho_4)_{00} = 1 + \frac{1}{\bar{\nu}\chi} \tag{3.16}$$

$$\Phi_{00}^i = \mathbf{e}_i, i = 1, 2, 3, \qquad \Phi_{00}^4 = \begin{pmatrix} \frac{4}{3}\frac{1}{2\,\mathrm{Pr}} + \left(\frac{\gamma-1}{2\,\mathrm{Pr}} - \frac{2}{3}\right)\alpha_3\chi \\ 0 \\ \frac{\gamma-1}{\mathrm{Pr}}\left(\alpha_1\chi - \frac{4}{3}\right) \\ \left(\frac{2}{3} - \alpha_1\chi\right)\left(\alpha_3\chi - \frac{1}{\mathrm{Pr}}\right) + \frac{\gamma-1}{2\,\mathrm{Pr}}\alpha_3\chi \end{pmatrix}, \tag{3.17}$$

χ denoting the strictly negative root of the algebraic equation (2.37). The matrices N_{00} and $\mathcal{A}_{00}$ are given by

$$N_{00} = \begin{pmatrix} 1 & 0 & 0 & \frac{(2/3 - (\gamma-1)/2\,\mathrm{Pr})\alpha_3\chi - (4/3)(1/2\,\mathrm{Pr})}{\det M} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{((\gamma-1)/\mathrm{Pr})(4/3 - \alpha_1\chi)}{\det M} \\ 0 & 0 & 0 & \frac{1}{\det M} \end{pmatrix} \tag{3.18}$$

and

$$\mathcal{A}_{00} = \begin{pmatrix} 1 & 0 & 0 & \frac{(4/3)(1/(2\,\mathrm{Pr}))(1/\chi) + ((\gamma-1)/(2\,\mathrm{Pr}) - 2/3)\alpha_3}{\bar{\nu}\det M} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{((\gamma-1)/\mathrm{Pr})(\alpha_1 - 4/3\chi)}{\bar{\nu}\det M} \\ 0 & 0 & 0 & 1 + \frac{1}{\bar{\nu}\chi} \end{pmatrix} \tag{3.19}$$

with

$$\det M = \left(\frac{2}{3} - \alpha_1 \chi\right)\left(\alpha_3 \chi - \frac{1}{\mathrm{Pr}}\right) + \frac{\gamma - 1}{2\,\mathrm{Pr}} \alpha_3 \chi \tag{3.20}$$

and we obtain the boundary conditions

$$\left(d_{1,j}^{n+1}\right)_1 = \left(g_{0,j}^{n+1}\right)_1 + \frac{(4/3)(1/(2\,\mathrm{Pr}))(1/\chi) + ((\gamma - 1)/(2\,\mathrm{Pr}) - 2/3)\alpha_3}{\overline{\nu} \det M} \left(g_{0,j}^{n+1}\right)_4 \tag{3.21}$$

$$\left(d_{1,j}^{n+1}\right)_2 = \left(g_{0,j}^{n+1}\right)_2 \tag{3.22}$$

$$\left(d_{1,j}^{n+1}\right)_3 = \left(g_{0,j}^{n+1}\right)_3 + \frac{((\gamma - 1)/\mathrm{Pr})(\alpha_1 - 4/(3\chi))}{\overline{\nu} \det M} \left(g_{0,j}^{n+1}\right)_4 \tag{3.23}$$

$$\left(d_{1,j}^{n+1}\right)_4 = \left(1 + \frac{1}{\overline{\nu}\chi}\right) \left(g_{0,j}^{n+1}\right)_4 \tag{3.24}$$

which can be viewed as first order in space approximations of the continuous relations

$$\nu \frac{\partial w_1}{\partial x} = \frac{(4/3)(1/(2\,\mathrm{Pr}))(1/\chi)(\Delta x/\Delta t) + ((\gamma - 1)/(2\,\mathrm{Pr}) - 2/3)\overline{V}_1}{\det M} w_4, \tag{3.25}$$

$$\nu \frac{\partial w_2}{\partial x} = 0, \tag{3.26}$$

$$\nu \frac{\partial w_3}{\partial x} = \frac{((\gamma - 1)/\mathrm{Pr})(\overline{V}_1 + \overline{C} - (4/(3\chi))(\Delta x/\Delta t))}{\det M} w_4, \tag{3.27}$$

and

$$\nu \frac{\partial w_4}{\partial x} = \frac{1}{\chi} \frac{\Delta x}{\Delta t} w_4 \tag{3.28}$$

written at the boundary points (x_0, y_j) and at time t_{n+1}.

4. NUMERICAL RESULTS

4.1. *The 1D Case*

The numerical settings, which have been defined in Subsections 4.1 and 4.5 of reference [6], are partly recalled below to make the presentation self-sufficient.

The linearized 1D Navier–Stokes equations expressed in terms of the characteristic variables are solved in the segment [0, 1] of the Ox axis and we restrict ourselves to the case $0 < \overline{V} < \overline{C}$, where both the inflow and outflow boundaries are of subsonic type ($\overline{V}$ and $\overline{C}$ denote the linearized velocity and speed of sound, respectively). This case is more complex than the supersonic case $\overline{C} < \overline{V}$ because information propagates against the flow (see [6]).

At $x = 0$ and at $x = 1$, we successively adopt:

—the absorbing boundary conditions for the Euler equations [6],
—the continuous artificial boundary conditions of orders 0 and 1 with respect to ν [6],
—the discrete artificial boundary conditions of orders 0 and 1 with respect to ε.

We choose $\overline{V} = 1$, $\overline{\rho} = 1$, and $\overline{C} = 2$ and the classical values $\gamma = 1.4$ and $\mathrm{Pr} = 0.75$. The kinematic viscosity ν is set to 0.1. We also take $R = 1$ for the Mayer's constant as we assume that the equations have been non-dimensionalized. As $\overline{C}$ is related to $\overline{T}$ by $\overline{C} = (\gamma R \overline{T})^{1/2}$, we have $\overline{T} \cong 2.86$. The flow is subsonic and the characteristic variables w_1, w_2, and w_3 propagate at respective speeds $\overline{V}+\overline{C}=3$, $\overline{V}=1$, $\overline{V}-\overline{C}=-1$. Each characteristic variable has initial value

$$f_0(x) = \begin{cases} e^{1/r^2} e^{1/((x-x_C)^2-r^2)} & \text{if } |x-x_C|<r, \\ 0 & \text{otherwise,} \end{cases}$$

with $x_c = 1/2$ and $r = 1/4$. f_0 belongs to $C^\infty(\mathbb{R})$ and has compact support in the bowl centered around x_c with radius r.

The segment [0, 1] is divided into I intervals $[x_i, x_{i+1}]$, $0 \le i \le I-1$ with $I = 1/\Delta x$ and $x_i = i\,\Delta x$. We have chosen $\Delta x = 10^{-2}$, i.e., $I = 100$. The solution of the Cauchy problem is obtained from a computation on an interval $[-L, 1+L]$ with L "sufficiently" large (see [5] for details).

The stability condition used in the computations reads

$$\Delta t \le \frac{\Delta x^2}{(|\overline{V}|+\overline{C})\Delta x + 2\nu \max(4/3, \gamma/\mathrm{Pr})}$$

and is derived from a Von Neumann analysis applied to a model scalar advection-diffusion equation [5].

In Fig. 1, we have compared the errors corresponding to the absorbing boundary conditions for the Euler equations, the continuous artificial boundary conditions of orders 0 and 1 with respect to ν, and the discrete artificial boundary conditions of orders 0 and 1 with respect to ε. We can see that there is little difference between the discrete and continuous artificial boundary conditions.

The discrete artificial boundary conditions can also be used as numerical boundary conditions in the case where the continuous artificial boundary conditions need to be completed. As explained in [6], when the number of boundary conditions is less than the number of unknowns, it is necessary to introduce extra relations, the so-called "numerical boundary conditions," in order to close the system that has to be solved on the boundary. We know that at a subsonic outflow boundary, the unique absorbing boundary condition for the Euler equations needs to be completed by two numerical boundary conditions whereas only one numerical boundary condition is necessary for the two continuous artificial boundary conditions [5, 6]. In Ref. [6], we have used upwind discretizations of the advection equations for the outgoing characteristic variables. Figure 2 shows the results obtained when replacing these extra relations by the corresponding discrete artificial boundary conditions of order 1 with respect to ε. At the inflow boundary ($x=0$), we impose the transparent boundary conditions for the Euler equations completed with an upwind discretization of the advection equation for the outgoing characteristic variable w_3. There is a slight improvement for the transparent boundary conditions for the Euler equations as well as for the continuous artificial boundary conditions of order 0 with respect to ν but no sensible difference for the continuous artificial boundary conditions of order 1 with respect to ν.

The discrete artificial boundary conditions have been built on the basis of a semi-discrete equation equivalent at second order in space to the explicit first order upwind scheme and for the two above results, the computations have been performed using the explicit first order upwind scheme. In order to study the behaviour of the discrete artificial boundary conditions when used in conjunction with other schemes, we have employed successively the Lax–Wendroff scheme and the flux corrected transport algorithm defined in [6, 7]. The results

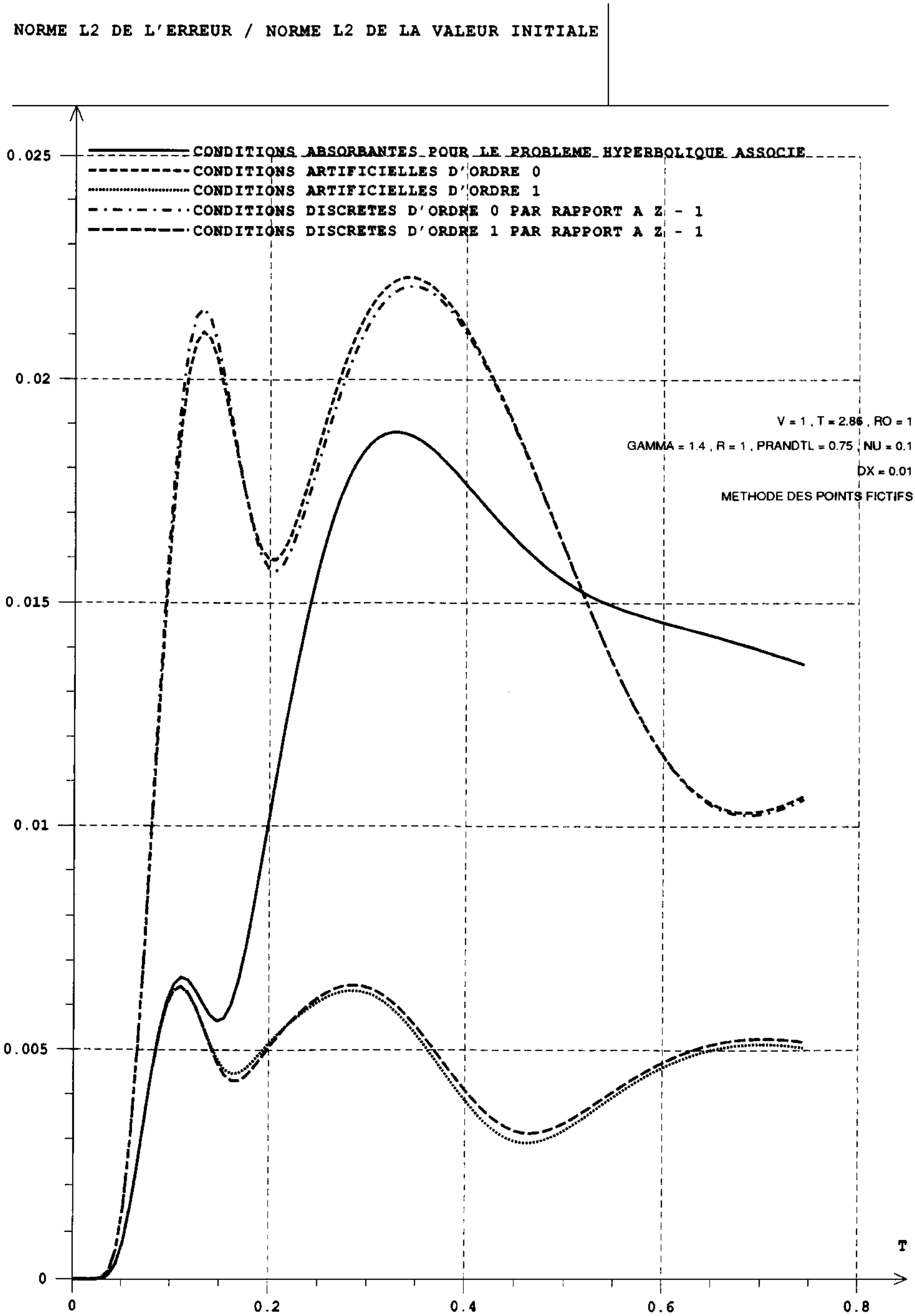

FIG. 1. Time-evolution of the (normalized) l^2-norm of the error between the solution of the Cauchy problem and the solutions of the mixed problems associated to the absorbing boundary conditions for the Euler equations (solid line), the continuous artificial boundary conditions of orders 0 (dashed line) and 1 (dotted line), and the discrete artificial boundary conditions of orders 0 (dot-dashed line) and 1 (long-dashed line).

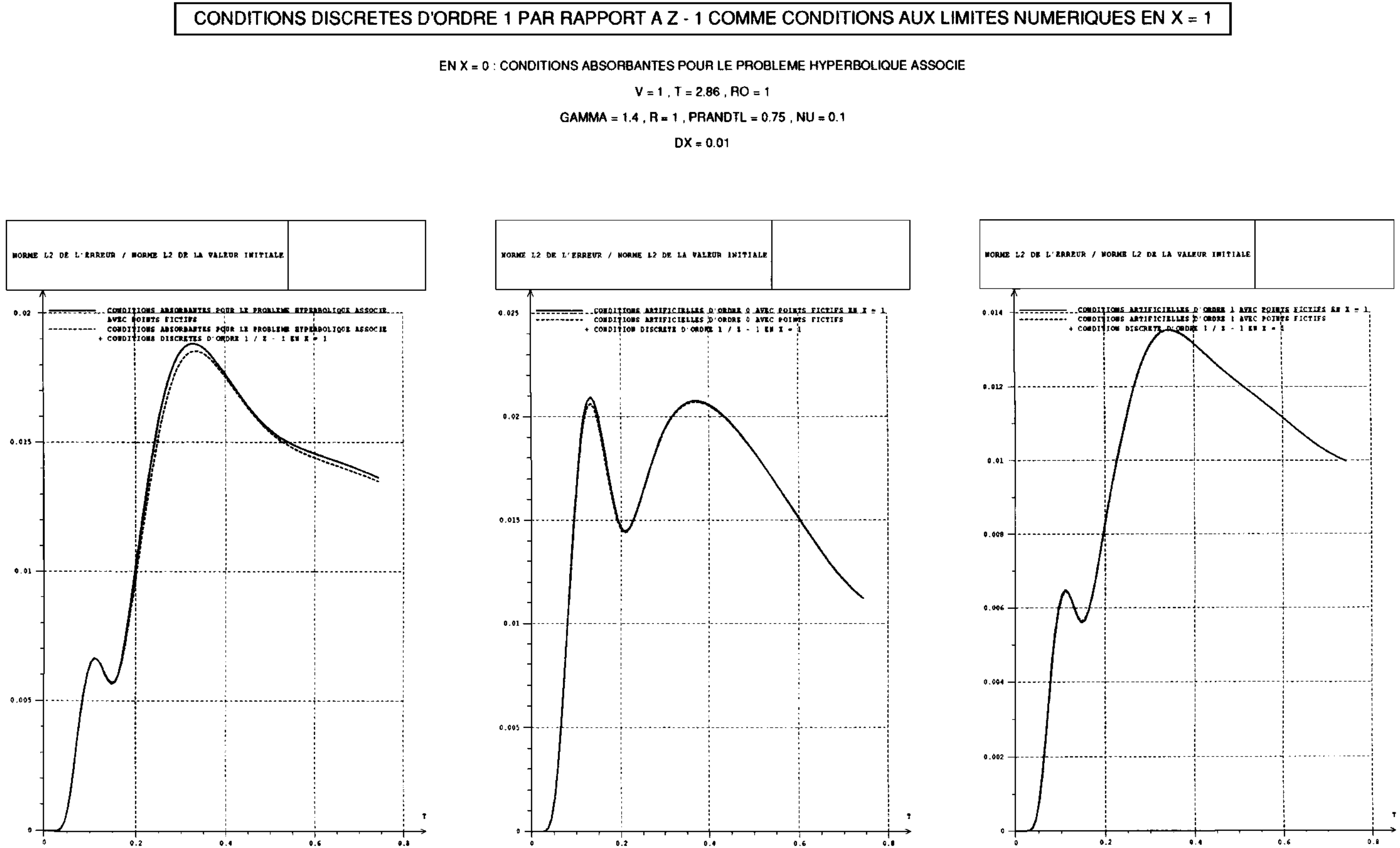

FIG. 2. Time-evolution of the l^2-norm of the error when the discrete artificial boundary conditions of order 1 (dashed line) are used as numerical boundary conditions at $x = 1$ instead of upwind discretizations of the outgoing characteristic variables (solid line) to complete the absorbing boundary conditions for the Euler equations (left), the continuous artificial boundary conditions of order 0 (center), and the continuous artificial boundary conditions of order 1 (right).

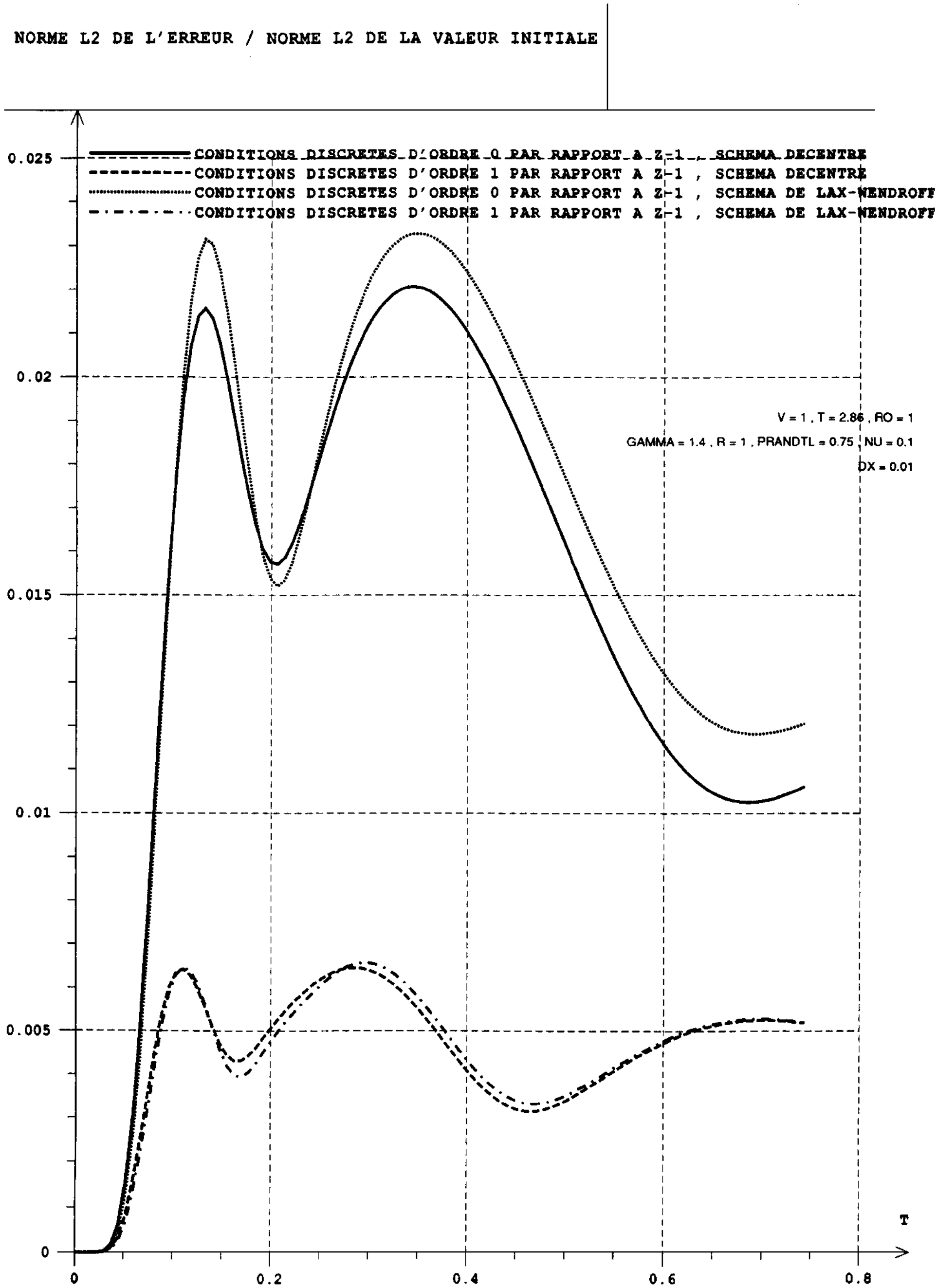

FIG. 3. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions when used in conjunction with the first order upwind scheme (solid line, order 0; dashed line, order 1) or with the Lax–Wendroff scheme (dotted line, order 0; dot-dashed line, order 1).

correspond to Figs. 3 and 4 and show that we can safely use the discrete artificial boundary conditions together with other schemes than the explicit first order upwind scheme. The artificial boundary conditions are applied at the first ficticious point for the Lax–Wendroff scheme and at the second one for the FCT algorithm.

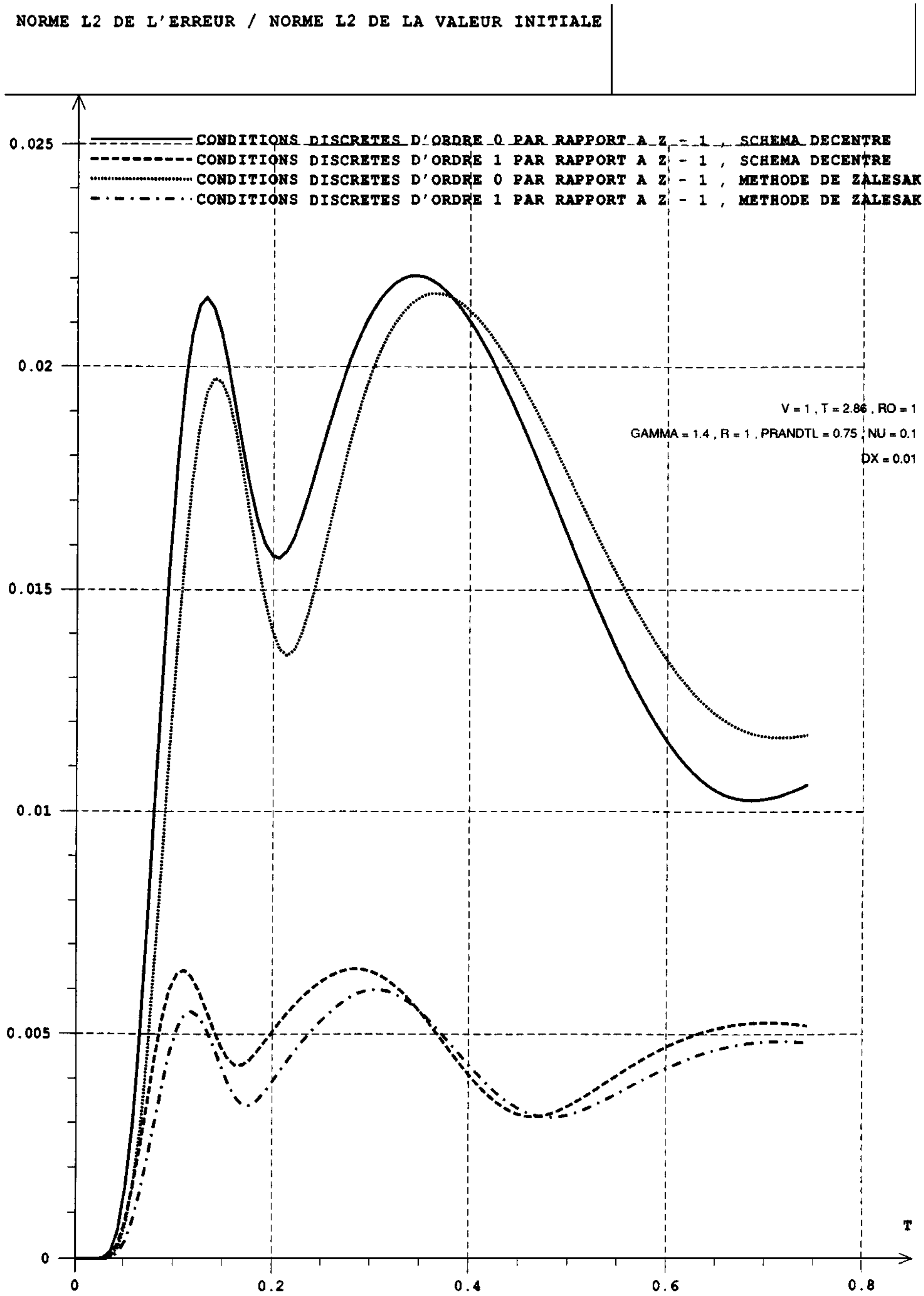

FIG. 4. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions when used in conjunction with the first order upwind scheme (solid line, order 0; dashed line, order 1) or with Zalesak's FCT algorithm (dotted line, order 0; dot-dashed line, order 1).

With the discrete approach, it is possible to approximate the discrete solution outside the computational domain at any number of ficticious points, thus allowing the use of schemes with five or even more points. For a five points scheme, for example, it is necessary to approximate the discrete solution also at the second ficticious point ($i = 2$). At first order with respect to ε, we have $\widehat{d_2} = (\mathcal{A}_0 + \varepsilon\mathcal{A}_1)\widehat{g_0}$, where the jth columns of the 3×3 matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ are respectively given by

$$(\mathcal{A}_0)_{\cdot j} = \sum_{i=1}^{3} (N_{ij})_0 (\rho_i)_0^2 \Phi_o^i \tag{4.1}$$

and

$$(\mathcal{A}_1)_{\cdot j} = \sum_{i=1}^{3} \left[(N_{ij})_0 (\rho_i)_0^2 \Phi_1^i + (N_{ij})_0\, 2(\rho_i)_0 (\rho_i)_1 \Phi_o^i + (N_{ij})_1 (\rho_i)_0^2 \Phi_o^i \right] \tag{4.2}$$

and we finally obtain

$$d_2^{n+1} = \left(\mathcal{A}_0 \mathcal{A}_1^{-1} + I\right) d_2^n - \mathcal{A}_0 \mathcal{A}_1^{-1} \mathcal{A}_0 g_0^n. \tag{4.3}$$

Figure 5 presents the results obtained with the FCT algorithm when applied without any modification up to the boundary.

The discrete artificial boundary conditions being based on the first order upwind discretization, applying them directly to a high order five-node scheme, as done here, may keep the overall order of approximation low. When the main interest is accuracy, then the construction of similar boundary conditions on the basis of a high order scheme should be considered. It would require considering a higher dimension eigenvalue problem instead of (1.20)–(1.21), with the summation with respect to α extending from -2 to 2 rather than from -1 to 1. This would introduce additional roots ρ and additional eigenvectors which, in turn, would have to be taken into account in the analysis of Section 3.

There are situations, however, where just knowing how to define the discrete solution outside the computational domain is more important than accuracy, thus justifying applying the present low-order artificial boundary conditions to a (potentially) high-order interior scheme. An example of such a situation is the method developed by Jameson *et al*. [8], which is widely used with great success in many industrial CFD codes. It is a second order finite volume scheme with a central differencing of the fluxes, in which third order additional dissipation terms are added to control the damping of high frequency waves. These numerical damping terms require the evaluation of third differences at cell interfaces. At an artificial boundary, values of the solution have to be prescribed at two nodes outside the computational domain. Besides the standard approaches [9], which are more focused on the damping properties than on the overall accuracy, the present approach provides a mean of defining "exterior" values which are closer to the physics, as they approximate the discrete solution outside the computational domain.

4.2. *The 2D Case*

We recall below the model problem described in Subsection 5.1 of Ref. [6].

We want to solve the linearized 2D Navier–Stokes equations on the strip $\mathbb{R} \times [0, 1]$ of the xOy plane.

At $x = 0$ and at $x = 1$, we introduce artificial boundaries where we successively adopt:

—the continuous artificial boundary conditions of orders (0, 0), (1, 0), and (1, 1) with respect to $(\nu, i\eta/s)$ [6],

—the discrete artificial boundary conditions of orders (0, 0), (1, 0), and (1, 1) with respect to $(\varepsilon, \eta^* = \eta\Delta y/\varepsilon)$.

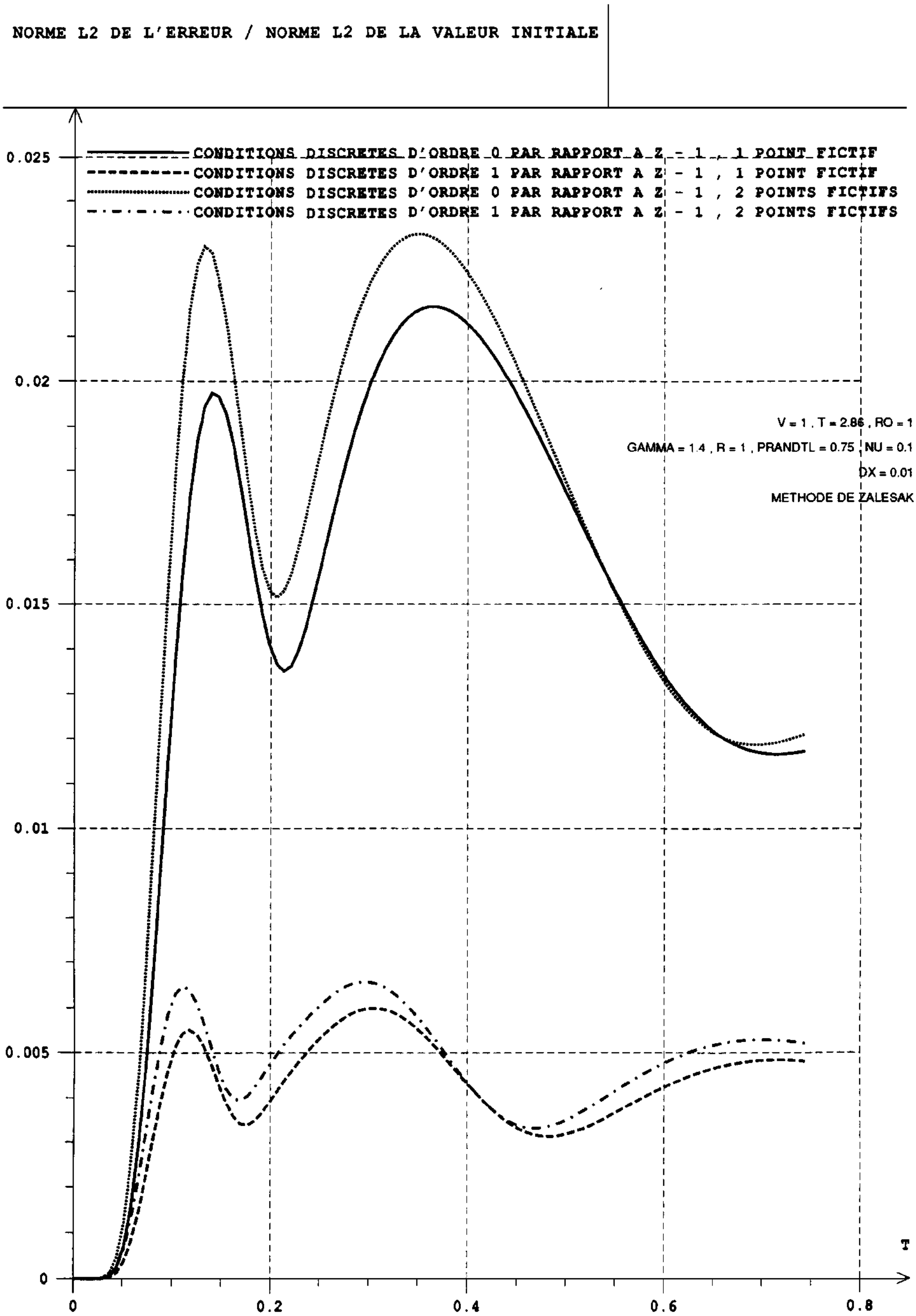

FIG. 5. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions applied at one ficticious point (solid line, order 0; dashed line, order 1) or two ficticious points (dotted line, order 0; dot-dashed line, order 1).

On the north boundary ($y = 1$), we impose in all cases the absorbing boundary conditions of order 0 for the Euler equations. On the south boundary ($y = 0$), we also employ the absorbing boundary conditions of order 0 for the Euler equations except when $\overline{V_2} = 0$ and $u(., t = 0)$ is symmetrical with respect to the Ox axis which becomes a symmetry axis.

For quantities $\overline{V}_1$, $\overline{T}$, $\overline{\rho}$, ν, γ, R, and Pr, we keep the values of Subsection 4.1. Moreover, we choose $\overline{V}_2 = 0$. As $\overline{V}_1 < \overline{C}$ and $\overline{V}_2 < \overline{C}$, the flow is subsonic in each space direction. The west boundary is of subsonic inflow type whereas the east boundary is of subsonic outflow type. Let us introduce the two scalar functions

$$f_0(x, y) = \begin{cases} e^{1/r^2} e^{1/((x-x_C)^2+(y-y_C)^2-r^2)} & \text{if } (x - x_C)^2 + (y - y_C)^2 < r^2, \\ 0 & \text{otherwise} \end{cases}$$

$$g_0(x, y) = f_0(x, y) \cos[k_x(x - x_C) + k_y(y - y_C)],$$

where $f_0 \in C^\infty(\mathbb{R}^2)$ has compact support in the bowl centered around point (x_C, y_C) with radius r.

The initial value is defined by $\tilde{V}_2 = 0$ and $\tilde{V}_1 = \tilde{T} = \frac{\tilde{\rho}}{\overline{\rho}} = f_0$, or $\tilde{V}_1 = \tilde{T} = \frac{\tilde{\rho}}{\overline{\rho}} = g_0$. By modifying the direction of wave vector $\mathbf{k} = (k_x, k_y)^t$ in function g_0, we can study the effects of the approximation with respect to the parameter η^*.

The segment [0, 1] is divided into $I = 1/\Delta x$ intervals $[x_i, x_{i+1}]$, $0 \le i \le I - 1$ on the Ox axis and $J = 1/\Delta y$ intervals $[y_j, y_{j+1}]$, $0 \le j \le J - 1$ on the Oy axis. We have chosen $\Delta x = \Delta y = 2\ 10^{-2}$, i.e., $I = J = 50$.

The numerical scheme is Zalesak's FCT algorithm [6, 7].

In Fig. 6, we have superimposed the error curves associated to the discrete artificial boundary conditions of orders (0, 0), (1, 0), and (1, 1) when the initial value of the solution is the function f_o with $x_c = 1/2$, $y_c = 0$, and $r = 1/4$.

Figure 7 allows us to study the behaviour of the discrete artificial boundary conditions of order (1, 1) for different values of the angle between the x axis and the vector $\mathbf{k}$ in the function g_o with $\|\mathbf{k}\| = \frac{2\pi}{10\Delta x}$, $x_C = y_C = 1/2$, $r = 0.45$. As for the continuous artificial boundary conditions, we observe for long times that the error decreases with the angle of incidence, which is coherent with the approximations made.

Figure 8 compares the discrete and continuous artificial boundary conditions of orders (0, 0), (1, 0), and (1, 1). For the orders (0, 0) and (1, 0), we obtain very similar results whereas for the order (1, 1) the continuous artificial boundary conditions give the best results.

5. HIGHER ORDER DISCRETE ARTIFICIAL BOUNDARY CONDITIONS

We have seen that the discrete and continuous artificial boundary conditions of orders (0, 0) and (1, 0) produce very similar results. On the other hand, the continuous artificial boundary conditions of order (1, 1) produce a lower error than the corresponding discrete artificial boundary conditions and we will therefore try to improve them by taking into account the terms in $\varepsilon\eta^* = \eta\Delta y$ in the approximation of formula (1.22). The first step consists in completing the asymptotic expansions of the generalized eigenvalues and eigenvectors.

5.1. *Generalized Eigenvalues and Eigenvectors of the First Kind*

Setting $\Phi = \Phi_{00} + i\eta^*\Phi_{01} + \varepsilon\Phi_{10} + \varepsilon i\eta^*\Phi_{11} + O(\varepsilon^2) + O(\eta^{*2})$, the vector Φ_{11} is solution of the linear system $\mathcal{M}_{10}\Phi_{11} = -(\mathcal{M}_{01}\Phi_{10} + \mathcal{M}_{20}\Phi_{01} + \mathcal{M}_{11}\Phi_{00}) = b$. It requires the

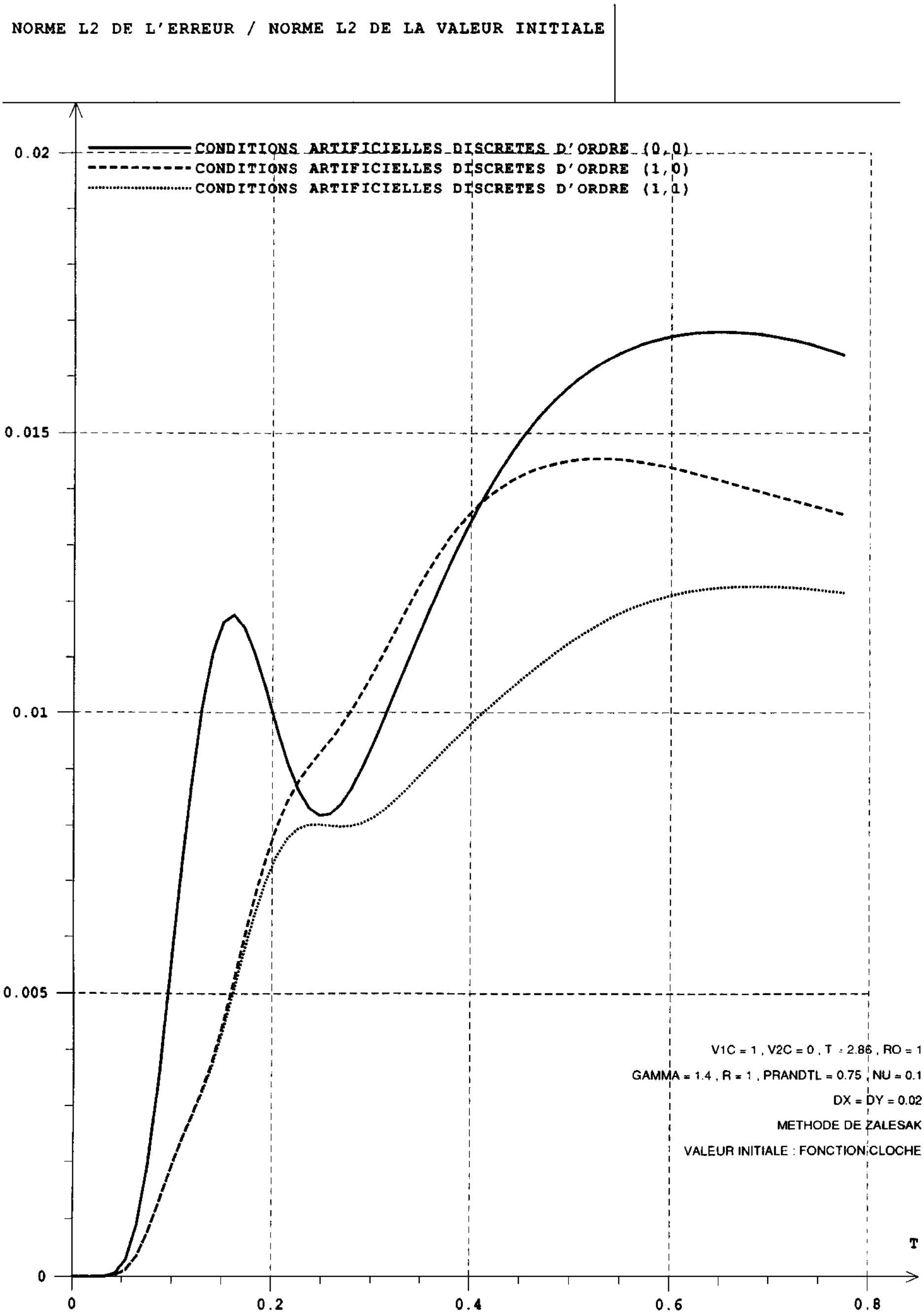

FIG. 6. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions of orders (0, 0) (solid line), (1, 0) (dashed line), and (1, 1) (dotted line) with function f_0 as initial value.

determination of ρ_{11}, which is given by $\det'(\mathcal{M}_{10}) \cdot \mathcal{M}_{11} = 0$, and we obtain

$$(\rho_i)_{11} = \frac{2\overline{V}_2(\Delta t/\Delta y)}{\alpha_i^2}\left(1 + \frac{\alpha_i^- + \overline{\nu} B_{ii}}{\alpha_i}\right), \qquad i = 1, \ldots, 4.$$

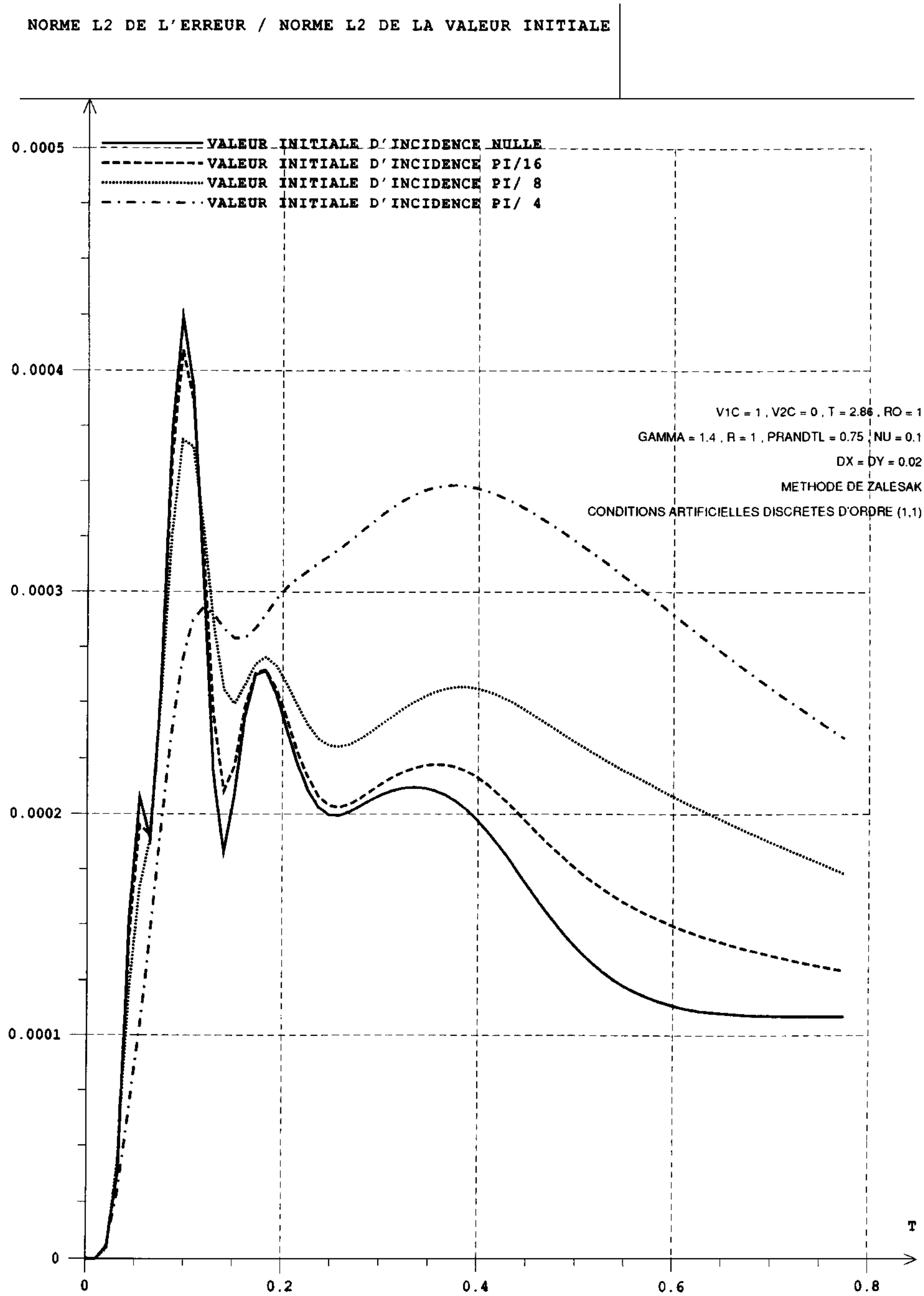

FIG. 7. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions of order (1, 1) for $(\widehat{Ox, \mathbf{k}}) = 0$ (solid line), $\pi/16$ (dashed line), $\pi/8$ (dotted line), and $\pi/4$ (dot-dashed line) in function g_0.

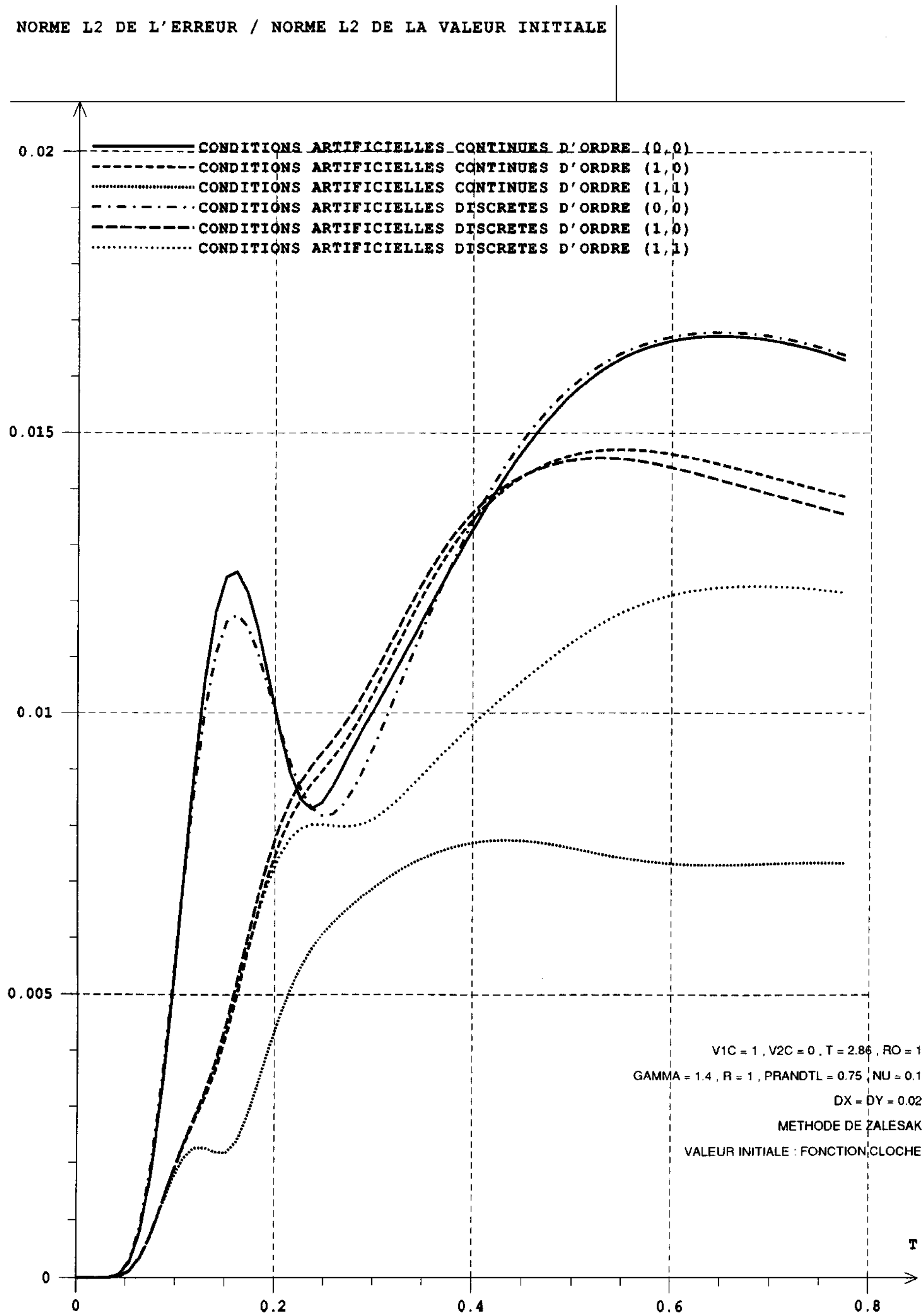

FIG. 8. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions of orders (0, 0) (dot-dashed line), (1, 0) (long-dashed line), and (1, 1) (dotted line) compared to the continuous artificial boundary conditions of orders (0, 0) (solid line), (1, 0) (dashed line), and (1, 1) (dotted line).

Φ_{11} is then given by

$$\Phi_{11} = \left(0, \frac{\alpha_1 b_2}{\alpha_2 - \alpha_1}, \frac{\alpha_1 b_3}{\alpha_3 - \alpha_1}, \frac{\alpha_1 b_4}{\alpha_4 - \alpha_1}\right)^t \qquad \text{for } i = 1,$$

$$\Phi_{11} = \left(\frac{\alpha_2 b_1}{\alpha_1 - \alpha_2}, 0, 0, \frac{\alpha_2 b_4}{\alpha_4 - \alpha_2}\right)^t \qquad \text{for } i = 2,$$

$$\Phi_{11} = \left(\frac{\alpha_3 b_1}{\alpha_1 - \alpha_3}, 0, 0, \frac{\alpha_3 b_4}{\alpha_4 - \alpha_3}\right)^t \qquad \text{for } i = 3,$$

and

$$\Phi_{11} = \left(\frac{\alpha_4 b_1}{\alpha_1 - \alpha_4}, \frac{\alpha_4 b_2}{\alpha_2 - \alpha_4}, \frac{\alpha_4 b_3}{\alpha_3 - \alpha_4}, 0\right)^t \qquad \text{for } i = 4.$$

5.2. *Generalized Eigenvalues and Eigenvectors of the Second Kind*

ρ_{01} is solution of $\det'(\mathcal{M}_{00}) \cdot \mathcal{M}_{01} = 0$ and is given by

$$\rho_{01} = -\frac{\det'(\mathcal{D}_1/\overline{\nu} - \chi \mathcal{D}_0 + B) \cdot \left[\rho_{00}(\Delta t/\Delta y)\mathcal{A}^{(2)} - \left(1 - \rho_{00}^2\right)(\nu \Delta t/(\Delta x \Delta y)) B^{(1,2)}\right]}{\det'(\mathcal{D}_1/\overline{\nu} - \chi \mathcal{D}_0 + B) \cdot \mathcal{D}_0}.$$

With the notation $\Phi = \Phi_{00} + \varepsilon \Phi_{10} + i\overline{\eta}\Phi_{11} + O(\varepsilon^2) + O(\overline{\eta}^2)$, Φ_{11} is solution of the linear system $\mathcal{M}_{00}\Phi_{11} = -\mathcal{M}_{01}\Phi_{00}$ or equivalently $[\mathcal{D}_0 + (1 - \rho_{00})(\mathcal{D}_1 + \overline{\nu} B)]\Phi_{11} = b$ where we have set

$$b = \left[\rho_{01}(\mathcal{D}_1 + \overline{\nu} B) + \overline{\nu}\chi\rho_{00}\frac{\Delta t}{\Delta y}\mathcal{A}^{(2)} + (1 + \rho_{00})\frac{\nu \Delta t}{\Delta x \Delta y} B^{(1,2)}\right]\Phi_{00}. \tag{5.1}$$

For $\chi = \chi_i$, $i = 1, 2, 3$, Φ_{11} has the same expression as Φ_{10} in (2.44), with b and Δ defined by (5.1) and (2.46), respectively. For $\chi = \chi_4$, we denote X_1, X_2, X_3, and X_4 as the coordinates of vector Φ_{11}. We then have $X_2 = 0$, whereas X_1, X_3, and X_4 solve the 3×3 linear system with invertible matrix

$$\begin{pmatrix} \alpha_1 + (1-\rho_{00})(\alpha_1^- + \overline{\nu} B_{11}) & (1-\rho_{00})\overline{\nu} B_{13} & (1-\rho_{00})\overline{\nu} B_{14} \\ (1-\rho_{00})\overline{\nu} B_{31} & \alpha_3 + (1-\rho_{00})(\alpha_3^- + \overline{\nu} B_{33}) & (1-\rho_{00})\overline{\nu} B_{34} \\ (1-\rho_{00})\overline{\nu} B_{41} & (1-\rho_{00})\overline{\nu} B_{43} & \alpha_4 + (1-\rho_{00})(\alpha_4^- + \overline{\nu} B_{44}) \end{pmatrix}$$

and right hand side

$$\overline{\nu}\chi_4\rho_{00}\frac{\Delta t}{\Delta y}\begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + (1 + \rho_{00})\frac{\nu \Delta t}{\Delta x \Delta y}\frac{1}{6}\begin{pmatrix} \frac{1}{2C} \\ 0 \\ \frac{1}{2C} \end{pmatrix}.$$

5.3. *Higher Order Discrete Artificial Boundary Conditions*

We now have all the necessary elements to proceed to the evaluation of the terms in $\varepsilon\eta^*$. Let the matrix M introduced in Subsection 1.3 be expanded as

$$M = M_{00} + i\eta^* M_{01} + \varepsilon M_{10} + \varepsilon i \eta^* M_{11} + O(\varepsilon^2) + O(\eta^{*2}).$$

The term in $\varepsilon i\eta^*$ in the corresponding development of its inverse N is given by

$$N_{11} = -N_{00}M_{11}N_{00} + N_{00}M_{01}N_{00}M_{10}N_{00} + N_{00}M_{10}N_{00}M_{01}N_{00},$$

the following term has to be added to the right-hand side in (3.2),

$$\begin{aligned}\varepsilon i\eta^* \sum_{j=1}^{4}(\widehat{g_0})_j \sum_{i=1}^{4} \big[&(N_{ij})_{11}(\rho_i)_{00}\Phi^i_{00} + (N_{ij})_{00}(\rho_i)_{01}\Phi^i_{00} \\ &+ (N_{ij})_{00}(\rho_i)_{00}\Phi^i_{11} + (N_{ij})_{00}(\rho_i)_{10}\Phi^i_{11} + (N_{ij})_{01}(\rho_i)_{10}\Phi^i_{00} \\ &+ (N_{ij})_{01}(\rho_i)_{00}\Phi^i_{10} + (N_{ij})_{10}(\rho_i)_{00}\Phi^i_{11}\big] = \varepsilon i\eta^*\mathcal{A}_{11}\hat{g}_0,\end{aligned}$$

and (3.3) becomes

$$\widehat{d_1} = (\mathcal{A}_{00} + i\eta^*\mathcal{A}_{01} + \varepsilon\mathcal{A}_{10} + \varepsilon i\eta^*\mathcal{A}_{11})\widehat{g_0} + O(\varepsilon^2) + O(\eta^{*2}). \tag{5.2}$$

From relation (5.2), the expressions of $\varepsilon\widehat{d_1}$, $d_1^{n+1}(y)$, and $d_{1,j}^{n+1}$ are obtained by adding the terms

$$\varepsilon i\eta^*\left(\mathcal{A}_{01} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{11}\right)\widehat{g_0},$$

$$\left(\mathcal{A}_{01} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{11}\right)\Delta y\frac{d}{dy}\left(g_0^n - g_0^{n-1}\right)(y)$$

and

$$\left(\mathcal{A}_{01} - \mathcal{A}_{00}\mathcal{A}_{10}^{-1}\mathcal{A}_{11}\right)\frac{1}{2}\left(g^n_{0,j+1} - g^n_{0,j-1} - g^{n-1}_{0,j+1} + g^{n-1}_{0,j-1}\right)$$

to the right-hand sides of relations (3.13), (3.14), and (3.15), respectively.

Figure 9 compares the errors associated to the discrete artificial boundary conditions of order (1, 1), the high order discrete artificial boundary conditions, and the continuous artificial boundary conditions of order (1, 1) and we see that the high order discrete artificial boundary conditions have an intermediate position between the continuous artificial boundary conditions of order (1, 1), with an almost identical behaviour for $t \leq 0.2$, and the discrete artificial boundary conditions of order (1, 1) that they reach for $t \geq 0.35$.

6. CONCLUSION

A hierarchy of discrete artificial boundary conditions has been proposed for the compressible Navier–Stokes equations linearized about a constant state. As for the continuous approach, the method of derivation uses the Fourier and Laplace transforms together with asymptotic expansions under the assumptions of low time frequencies and long space wavelengths.

Unlike the continuous artificial boundary conditions, the discrete artificial boundary conditions are self-sufficient: their number is always equal to the number of unknowns. Moreover, they constitute a good source of numerical boundary conditions in the case where the continuous artificial boundary conditions need to be completed.

The discrete artificial boundary conditions have been implemented in 1D and 2D model problems and they compare quite well with the continuous artificial boundary conditions.

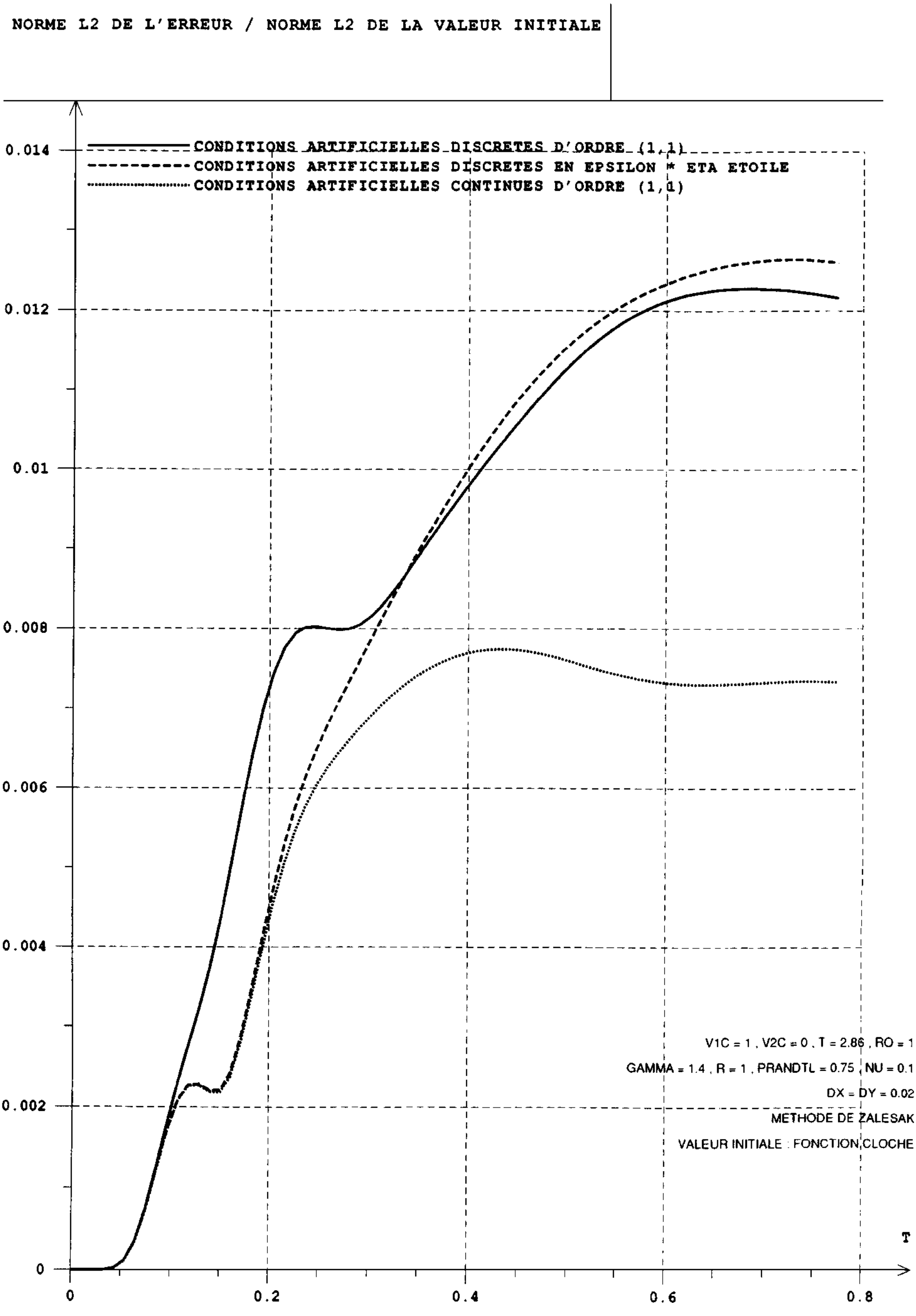

FIG. 9. Time-evolution of the l^2-norm of the error associated to the discrete artificial boundary conditions of order (1, 1) (solid line), the high order discrete artificial boundary conditions (dashed line), and the continuous artificial boundary conditions of order (1, 1) (dotted line).

Moreover, they can be coupled to schemes having arbitrary stencils, since they are based on the definition of the ficticious values of the discrete solution outside the computational domain.

A higher order discrete artificial boundary condition has also been proposed which improves the discrete artificial boundary conditions of order (1, 1) for the small times.

The discrete approach developed herein was based on a fully discrete scheme for the solution outside the computational domain, namely the first order upwind scheme. One very interesting point would be to consider a scheme semi-discrete in space, in order to have access to other time integration schemes (Runge–Kutta schemes, implicit schemes). For accuracy purposes, it would also be quite interesting to apply the method to higher order space discretizations in order to obtain discrete artificial boundary conditions which could be coupled to high-order interior schemes without degrading the overall accuracy.

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